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VI. *The Theory of the Singing Flame.* By E. G. RICHARDSON, *B.Sc., East London College, University of London.*

ABSTRACT.

The various theories of the action of the Singing Flame put forward since its discovery are reviewed, and experiments to test the relative merits of the two most recent theories are described ; Lord Rayleigh's theory is shown to fit the results most closely, in that (1) heat is given by the flame to the air in the tube at each condensation, and (2) stationary waves are formed in the gas as well as in the air-tube. But the lengths of gas-tube unfavourable to the " singing " cover, in fact, a much more restricted range than Lord Rayleigh surmised.

THE fact that a jet of hydrogen, burning in an open tube, would, under certain conditions, cause a musical note to be emitted, was first observed by HIGGENS* in 1777, and other observers studied various aspects of the phenomenon, without attempting to account for it, until De La Rive advanced his theory, *i.e.*, that the periodic condensation of water vapour by the burning hydrogen caused the emission of the note. That this cannot be the true cause of the " singing " was shown by Faraday, who was able to replace the hydrogen by carbonic oxide, a gas which produces no moisture in its combustion, without detriment to the effect. Faraday himself put forward an alternative theory, that the note was caused by successive explosions of the gas with the oxygen of the air, the flame then dying out until a further supply of air arrived at the jet, when it was re-ignited. For Wheatstone had shown in his revolving mirror that the flame was not steady, but vibrated up and down, so as to appear in the moving mirrors as a succession of images. As a result of his experiments, Sondhauss,† who of all the investigators has examined the phenomenon the most thoroughly, considered the cause of the " singing " to be the heating of the air in the neighbourhood of the jet, the subsequent change of density causing a compression to flow away from the jet, thus starting the air in the large outer tube (hereafter referred to as the air tube), to sound its natural tone. He also found that certain combinations of air tube and gas-supply tube would not sing, and concluded that the length of the gas tube must vary with different gases, in order that the oscillations of gas and air may be in step near the jet. If these oscillations were stopped by a plug of cotton-wool in the air tube near the jet, the singing ceased.

Zoch,‡ on the other hand, imagined that the current of air up the tube, caused by the combustion, was the origin of the singing. This idea was elaborated by Terquem§ as follows : the influx of air to the flame causes it to rise ; when it has used up the air, it falls again until fresh air rises, when the process is repeated, until after a few periods the rise and fall of the flame, and the periodic rise of air adjust themselves to the natural frequency of the air in the larger tube.

Lord Rayleigh‖ reverted to the heat transfer theory of Sondhauss. He showed theoretically that, unlike ordinary resonance, the impulses given to the

* Nicholson's Journ. (1797).
† Ann. der Phys. 109, 1 and 426 (1860).
‡ Ann. der Phys. 127, 589 (1866).
§ Ann. der Phys. 134, 468 (1868).
‖ Theory of Sound, II, 227.

resonator (*i.e.*, the air in the tube) by the impressing force (the heat) must occur at the maximum displacement or condensation, and not in the neutral position. Also that for the continuance of the vibrations the gas tube must be of such a length, and in such a position, that a condensation at the jet will travel down it and back again so as to arrive at the jet in phase with a condensation in the air tube (as Sondhauss had previously stated), in fact, that there should be stationary waves in the gas tube as well as in the air tube.

Finally, Gill,* apparently without having seen Lord Rayleigh's paper, has revived the older theory, wherein an accidental change of pressure in the air tube starts the latter faintly sounding, the successive condensations forcing the flame down into the jet, and each rarefaction letting it grow again.

It appears, then, that the modern theories divide on the question of the primum mobile of the action : does the air tube act as a resonator to the flame, or does the air in the tube force the movement of the flame into agreement with it ? It was in an attempt to settle which theory was best supported by facts that the following work was undertaken.

PHASE OF HEAT SUPPLY.

It is noted above that Lord Rayleigh's theory requires the heat to be given to the air at or in the quarter-period preceding a condensation ; so that, if the flame gives most heat when it is at its maximum size, the flame should be longest, and the gas should come out most freely a little before the maximum pressure in the air tube near the jet (if one allows for the time taken by the actual transfer of heat).

The other theory, in Gill's form of it, requires that each phase of maximum pressure should force the flame down, or temporarily stop the flow of gas, which should be most free at a rarefaction.

The vital point, then, is the relative phase of the vibrations of the singing flame and of the air near the jet. This I found in the following manner. A brass tube, 60 cm. by 4 cm. was taken and a gas tube with a jet† placed in it in such a position that the " singing " started spontaneously and vigorously. A hole, 1 cm. diameter, was bored in the brass tube at this point, also a larger hole, over which a piece of glass was fixed by sealing-wax, to serve as a window through which the singing flame could be observed. Over the small hole a König manometric capsule was placed, the burner of which was brought right round in front of, and close to, the window, so that when lit this manometric flame appeared vertically beneath the singing flame. The jets were placed as close as possible to each other, one inside and one outside the air tube ; and all joints were carefully plugged, as a leakage of pressure is inimical to the " singing." Both flames were fed by coal gas, and when the inner flame was singing, the two were examined in a rotating mirror, and two sets of images were seen one above the other. This is an inconvenient method of observation, because (1) the images jump about, as the four mirrors do not succeed each other in the same vertical plane ; (2) when the mirror and the flames are still, the latter appear correctly over each other only in one position of the mirror, owing to parallax. In order to compare the flames in a better way

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* Sill. Journ. 4, 177 (1897).

† With the gas at a pressure of 4 in. of water, I found that the diameter of the jet must lie between 0·4 and 2 mm. With smaller jets the flames will not sing, with larger they are immediately extinguished.

they were looked at directly through a stroboscope (a rotating card with equidistant slits whose speed could be varied). This apparatus, originally applied by Töpler[*] to the singing flame alone (without the König flame), allows one to lengthen the apparent period of the flames at will. By this means it was seen that *they rose and fell almost simultaneously and in the same phase.* This definitely rules out Mill's theory.[†]

In order to have a record of this phase relation I next attempted a photograph.
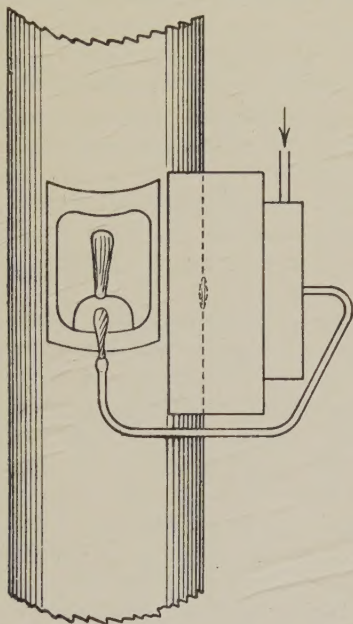

FIG. 1.

Manometric flames have been photographed several times, but not singing flames, nor, therefore, the two together. The following methods have been tried :—

(1) An instantaneous photograph of the revolving mirror.
(2) Sliding a sensitised plate across the back of the camera, while the flame is vibrating (Merritt[‡]).
(3) Keeping the plate fixed, but rotating the camera (Hallock[§]).

I tried all these three ways, but could get nothing from the first—indeed, I think no one has ever succeeded with this method ; nor from the second, and the third reproduced only the manometric flame, even when the gas was mixed with acetylene to render it more actinic, and with very rapid plates. The failure of the

[*] Ann. der Phys. 128, 126 (1866).
[†] The movement of the membrane is propagated with the speed of sound to the manometric flame ; the indications of the latter were consequently about 1/10,000 second " late " a time which can be neglected in comparison with the period of the oscillation (1/260 second).
[‡] Phys. Rev., 1, 166 (1893).
[§] Phys. Rev., 2 305 (1895).

singing flame to register seems to be due to its inferior luminosity compared to that of the manometric flame, added to the high frequency (about 260). I thereupon devised a new method which was more successful. The apparent period of the flames was lengthened by the stroboscope and they were photographed through the slits, the camera being rotated as in Hallock's method, using extra-rapid panchromatic plates, and feeding the flames with coal gas plus as much acetylene as they would bear without becoming smoky. The stroboscope allows impressions of several images of slightly differing phases to coincide and gives more chance to the light to act on the plate, but, of course, the stroboscope must be of slightly longer period, not shorter, otherwise the vibrations appear in the reverse order. At first the images were subject to parallax (Print 6), which only shows the true relationship of phase at the point marked ; but later, at the suggestion of Prof. Lees, parallax was avoided by rotating round the optical centre of the lens (Prints 9 to 13). Besides the phase relationship these prints also show the difference in character of the flames. The manometric flame rises to a maximum gradually and falls suddenly to start again, whereas the other rises quite suddenly, as Töpler noted, so that the heat is given over a small fraction of the period just before the maximum pressure.

In order to make this fit in rigorously with Lord Rayleigh's theory, we must suppose, either (1) that the heat is given instantaneously to the air when the flame is at its maximum, or (2) that the luminosity which accompanies the maximum is a sign of a fall in the temperature of the flame, which temperature is really greatest before the maximum height, and therefore a little before the compression occurs. In order to test this, I set up an optical pyrometer (Féry's) behind the stroboscope, and various phases were " held " and focused on the thermo-couple, but although the instrument showed differences in the amount of radiation it was not sufficiently sensitive to enable one to settle the question of which was the hottest ; perhaps a radio-micrometer would have done it. At any rate, the fact that, " with the same gas pressure, the temperature of a flame is greater as its surface is less " (Prof. Smithells, in the Dict. of Applied Chem.) is in favour of the second supposition.

With the stroboscope removed, it was found that a flame emitted much less energy when singing than the same flame when steady.

Gas.					Galvanometer Deflection.	
					Steady flame.	Singing flame.
Acetylene and coal gas	...	...	...	$9\frac{1}{2}$	....	3
Ditto	...	...	...	...	$7\frac{1}{4}$	$2\frac{1}{2}$
Ditto	...	...	...	...	6	2
Coal gas	...	...	...	...	$0\frac{1}{2}$	0

The Gas Supply Tube.

Lord Rayleigh in his Paper refers to the work of Sondhauss, stating that the latter found that the phenomenon depended largely on the length of the gas tube, and then proceeds to show that the tube must be shorter than a $\frac{1}{4}$-wave-length (for the particular gas and note emitted) or longer than a $\frac{1}{2}$ and shorter than $\frac{3}{4}$ of the wave-length in order that singing may start. Suitable data were therefore obtained, in order to test this question of the length of the gas tube. The results are best viewed on the accompanying graphs, each for a constant air tube, and showing the position in the air tube at which the flame due to each gas tube would

sing. Sondhauss used a gas tube which terminated in one of the tubules of a Wolff's bottle. I found that the bottle was not large enough to prevent variations of pressure at the lower end of the tube, and therefore a large earthenware jar of about 20 litres capacity was used, fed by coal gas from the main. The graphs show three distinct cases. In the case when the jet is slowly pushed up the air tube, the violence of the oscillation extinguishes the flame before it reaches the node of the air tube ; on the other hand, if the flame be placed in the upper half of the tube, and then moved down, it is extinguished before it reaches the node. These lengths correspond roughly to the lengths (less than $\frac{1}{4}L$——, between $\frac{1}{2}L$ and $\frac{3}{4}L$——, between L and $1\frac{1}{4}L$——, &c.) of Lord Rayleigh's theory, after allowance is made for the antinode at the lower end of the gas tube being a little distance beyond its end ; and for the node at its upper end lying considerably above the jet (*vide infra*). There are other comparatively restricted regions of the graph within which I have

Membrane   Gas   ←Gas   No Membrane

Gas→

FIG. 2.   FIG. 3.   FIG. 4.
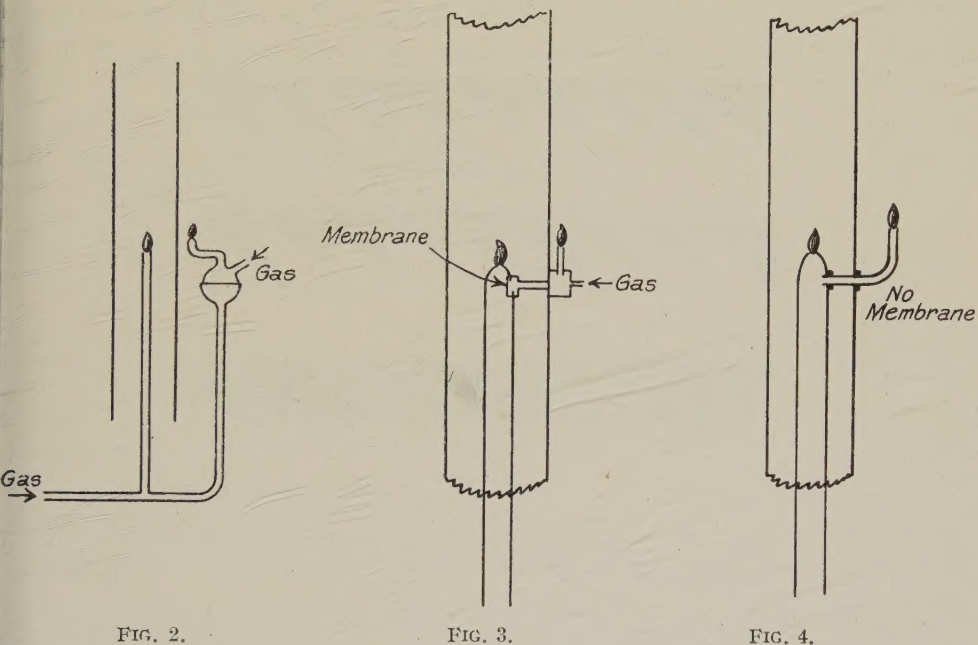
not been able to force the flame into vibration at all. These correspond roughly to lengths equal to or just greater than $\frac{1}{4}L$ and $\frac{3}{4}L$.

But it is not correct to say that the flame will not sing with gas tubes lying between these lengths. The singing is less intense, it is true, and often requires a larger flame encouraged by a tuning fork of the same pitch. Also the vibration is not intense enough to extinguish the flame at the node, where it goes on singing (Lord Rayleigh believed no singing possible at the node, and even beyond), until the changes of presssure in the air tube are too small to affect it, and it becomes steady.

It occurred to me that the explanation of these three different cases might be that in the first the gas tube was of such a length that it resounded in phase with

the air tube, but that in the second the vibration in the air tube, once started, was able to react on the flame, and to force the gas column to vibrate in spite of its natural period being different, just as a tuning fork can force a column of air into vibration, and, if the latter is of the correct length, to resound and assist the vibration. This was confirmed in the following manner.

Lord Rayleigh says "stationary or approximately stationary waves are formed in the gas as well as in the air." To test this I fixed a manometric capsule on a piece of glass tubing, 10 cm. by 2 cm., and made the gas tube of pieces of the same glass, so that the capsule could be "built in" to any place of the gas tube. I noticed that when the jet remained unlit, a tuning fork at the top of the air tube caused perturbations of the manometric on the gas tube. Accordingly, selecting lengths of air tube and gas tube, which produced vigorous singing, I was able to find well marked nodes and antinodes in the longer gas tube, corresponding fairly well with the value of λ, found from $V = n\lambda$, V for coal gas being 490 m./sec. (Zoch)[*] (Fig. 16). It will be noticed that, as mentioned above, the jet itself is not usually a node; this would be some distance above, near the node of the air tube. In
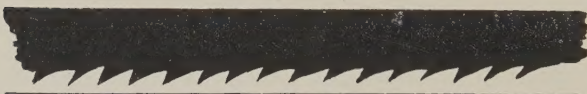
FIG. 5A.

FIG. 5B.

FIG. 5C.

order to test whether the large jar in which the tube ended was a true antinode, another manometric capsule was placed with its lead near the end of the gas tube, inside the jar.

When an example of the second case was tried—*i.e.*, air tube 63 cm., gas tube, 157 cm., no marked node or antinodes could be found, but the manometric flame was about equally indented, wherever it was placed in the gas tube, showing that here was an instance of forced vibration, but no resonance. Apropos of this state of affairs it may be noted that Rogers[†] succeeded in making an alcohol flame sing, even when the supply tube was plugged up by a wick to conduct the spirit; in this case there could be no resonant vibration of the tube.

In the third case mentioned above the gas tube was probably of such a length that a vibration sent down it and reflected back would arrive at the jet in the most inimical phase, and upset the vibration of the flame at once, so that "singing" once started would be immediately stopped (*vide* Lord Rayleigh, *loc. cit.*). How-

* Ann. der Phys., 128, 497 (1866).
† Phil. Mag. (4), 15, 261 (1858). Sill. Journal (2), 26, 1 (1858).
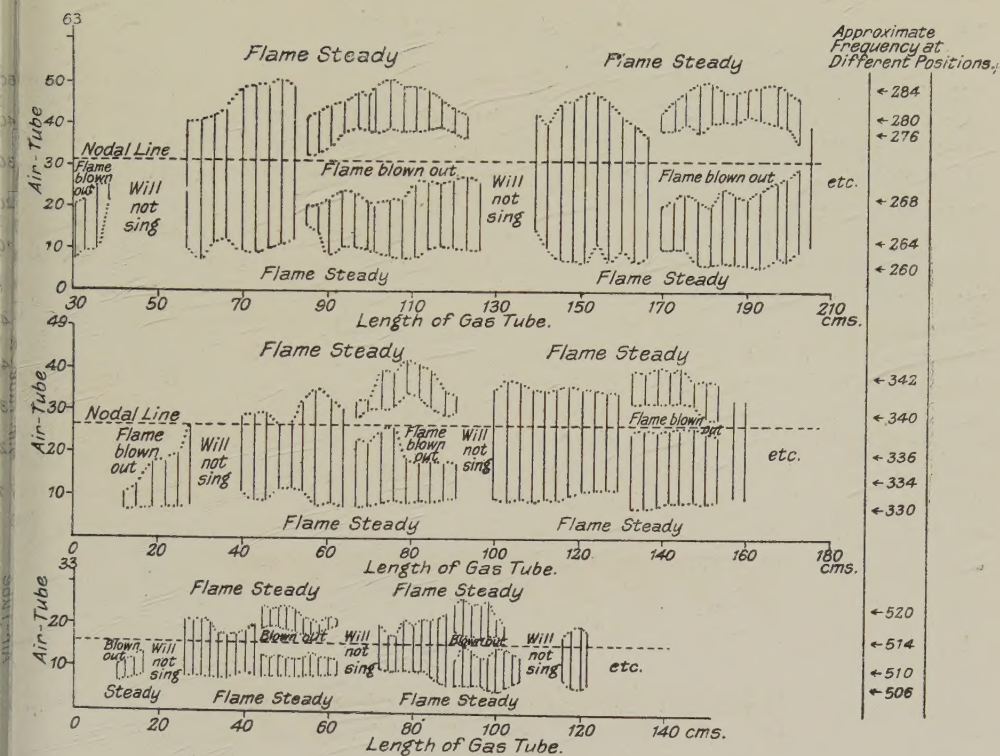
FIG. 15.

The vertical lines show the various positions in the air tube at which given gas tubes could be coaxed to sing—*e.g.*, the 5th line in the first graph shows that, with a gas tube 57 cm. long, the flame commenced to sing at a height of 9 cm. up the air tube, and continued so, when pushed further up, until it reached a height of 39 cm., when it ceased to sing.
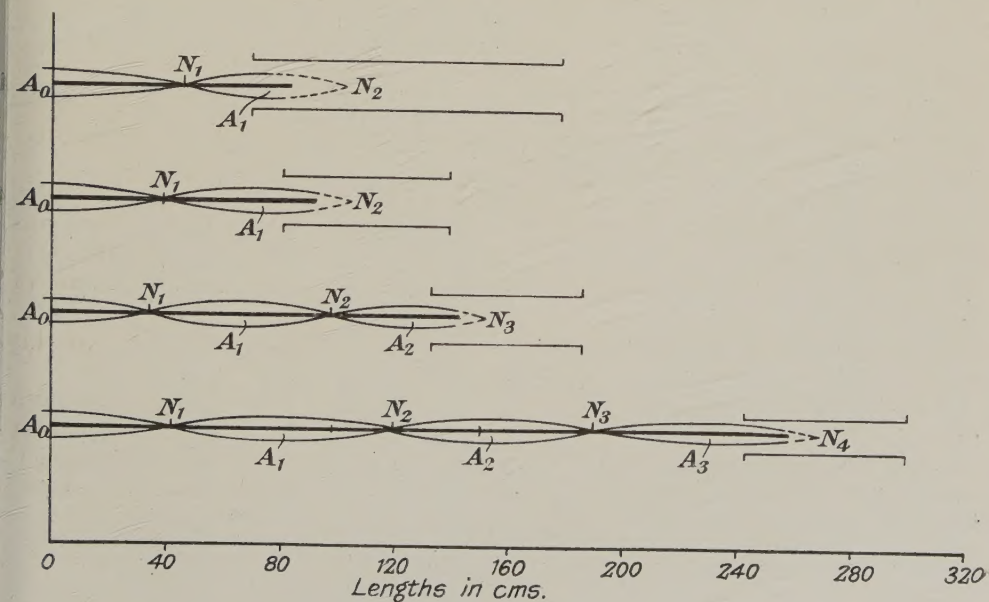

FIG. 16.

ever, I do not attach much importance to this gap ; in fact, Zoch says that every combination of tubes and gas which the ried could be made to act.

In order to find the relation between the pressure of the gas and the state of the flame, Gill attached a manometric flame to the base of the supply tube, and compared the flames in a revolving mirror (*see* Fig. 2). Before reading Gill's Paper I had tried to find the same relation by the apparatus shown below (Fig. 3), wherein a side tube led from the gas tube through the air tube, and so to a manometric capsule (the membrane may be dispensed with, and the gas lit directly in the side tube) (Fig. 4). I compared the flames in a mirror, but subsequently I repeated the observations with the more exact stroboscope.

All three forms of apparatus lead to the same result ; *the flames are in opposite phases.* This, as Gill truly stated, is due to the gas being held back when the flame strikes back to the jet, so momentarily increasing its pressure, as when a water tap is turned off. Gill considered that this showed that the gas was forced in by the air pressure, but adversely to this possibility the singing flame and air pressure are in phase. It follows that the air outside and the gas inside the jet are in opposite phase in regard to pressure.

CHANGES OF PITCH WITH DIFFERENT POSITIONS OF THE FLAME.

Two final points which I do not think have been noted before : As the jet is pushed up the air tube, the pitch of the singing flame rises. Observers have stated that the pitch of the note is the same as that natural to the tube, due regard being had to the temperature. In order to see whether this was so I lowered a taper into an organ pipe. On blowing the pipe by the bellows and raising the taper, the pitch was found to rise similarly, within the region wherein a flame will usually sing.

If a Singing Flame resumes the steady state in the upper half of the tube, just before it ceases, the frequency reverts to what it was initially. In corroboration of the change of pitch noted here, H. Pflaum, in his work on Rijke's Gauze, found a rise of more than a semitone as the heated gauze was moved up the tube.

HARMONICS.

It was early found that, in certain positions, a flame would give the fundamental of the air tube and its first harmonic at the same time. Such a flame, examined in the revolving mirror, showed smaller flames between the large ones (Fig. 5A). When the jet was pushed higher, only the big flames remained and the fundamental tone (Fig. 5B) ; when lower twice as many little flames (and the octave alone) were given (Fig. 5C). This shows that (in Fig. 5A) the large flames produce the fundamental and the two sets combined produce the harmonic. A steady photograph of such vibrating flames (Fig. 14) shows one flame within the other plainly.

In conclusion I wish to acknowledge my indebtedness to Professor Lees, of East London College, for suggesting this work, and for help in carrying it out.

DISCUSSION.

Prof. C. H. LEES (communicated) : While the Rayleigh theory of the flame covers most of the observed facts and may be taken as satisfactory in its broad outlines, there are other facts which Mr. Richardson has brought to light with which it is not in complete accord. To cover these some modification of the theory will be necessary, but so far as I can see these modifications are not of a serious nature.

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No. 13.

No. 14.

No. 9.

No. 12.

Nos. 7 and 8.
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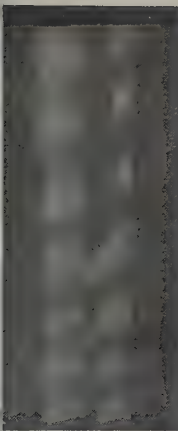
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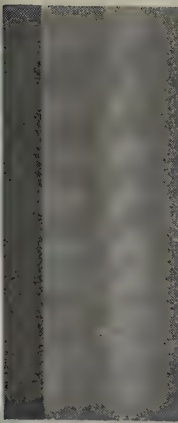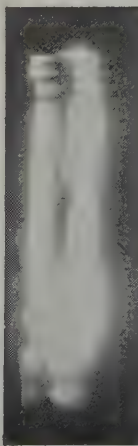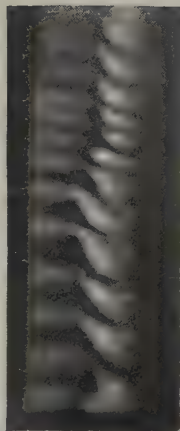
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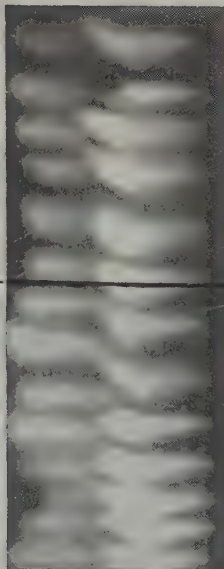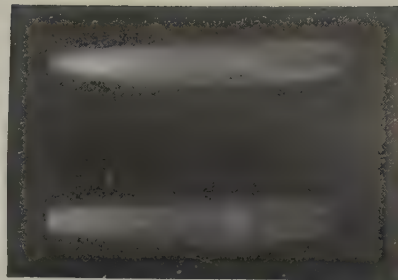
Figs. 6 to 14.—The Singing Flame is Above in Each Case. The Arrows Show the Direction in which the Flames Move.

VII. *Unit Surfaces of Cooke and Tessar Photographic Lenses.* By ALICE EVERETT,
M.A., The National Physical Laboratory.

ABSTRACT.

A number of rays in an axial plane (and a few general rays) are traced through
the lens systems by exact methods, and on each ray the positions of the conjugate points
for unit magnification are found by Mr. T. Smith's formulæ. The results show that for
general rays the loci of these " unit points " are three-dimensional. The loci are surfaces
only when the chief rays are bound by some conditions, such as passing through a fixed
point of the object. In this case the unit surfaces are not fixed, but shift with the object
point. The curvature changes with aperture, and with the distance of the object point
from the optic axis. Hence the unit surfaces cannot coincide with wave-fronts.
Within the region for which the lenses are designed, the curvature of both object and
image unit-point loci is positive (convex to the light source) and the image locus is more
curved than the object locus. These facts do not coincide with the theoretical conclusions
obtained by Mr. Smith for either a thin system or an ideal thick system, but accord
more nearly with the latter, especially as regards the sign of the difference of curvatures
of the two unit surfaces.

In this work :—

     The terms " unit points " and " unit surfaces " are used briefly to denote
         conjugate points for unit magnification, and their loci.

     The secondary focus (*i.e.*, focus for rays in a transverse section) only is
         considered.

     The suffix λ refers to the λ-th refracting surface, and medium following it.

     Distances are measured positively in the direction light travels.

WHILE engaged recently on some investigations relating to photographic lenses,
the writer was privileged to see an advance proof of an article by Mr. T. Smith
(" Optical Calculations," Dictionary of Applied Physics, Vol. 4, p. 287, 1922), in
which a passage occurs pointing out that the unit surfaces in a thin lens cannot
both be planes, and giving expressions for their curvatures in the simple cases of
(1) a single thin lens, when the rays are nearly parallel to the axis ; and (2) a
system of lenses of zero thickness in contact, when the rays are incident near the
vertex.

     The idea struck her that it might be of practical interest to investigate the
forms of the unit surfaces in an actual photographic lens system, and two well-
known types, Cooke and Tessar, were selected for examination. They have
comparatively few (six and seven) refracting surfaces, and some of the requisite
data had already been calculated by the writer for another purpose.

     The constructional data for the Tessar, which was designed by Rudolph
and Wandersleb for Zeiss, are taken from " Zeitschrift für Instrumentenkunde,"
1907, pp. 78, 79. It consists of four lenses—hence the name—of which the last
two are cemented, and is said (*see* Turrière) to be intended for portraiture,
instantaneous work, and general amateur purposes. The data for the Cooke,
which refer to a patent of 1894, are taken from von Rohr's " Theorie des

photographischen Objectivs," 1899, p. 240. (The first thickness is given wrongly as 2·59 in Gleichen's " Lehrbuch der geometrischen Optik," 1902, p. 490.) This lens, designed by H. Dennis Taylor, of T. Cooke & Sons, and manufactured by Taylor, Taylor & Hobson, Ltd., is remarkable for combining efficiency with simplicity, being composed of three single separated lenses, the first and third made of the same glass. The figures, it should be noted, are for an early form. Although the constructional data (also given in Turrière's " Optique Industrielle," Vol. 1), for both lenses are probably sufficiently accurate to illustrate their general features, it should not be assumed that they justify any deduction regarding relative merits of actual lenses of these two popular types.

Unfortunately, since at least one of the unit surfaces is always virtual, it is not possible to examine them *in situ* by direct physical means.

The process adopted for this investigation was to trace by exact methods the paths of a number of rays with different inclinations and apertures through the system examined, and then to find the positions of the unit points on each ray. For descriptions of modern methods of tracing rays reference may be made to Mr. T. Smith's articles in the Proceedings of the Physical Society (Vol. 25, 1913, p. 239; Vol. 27, 1915, p. 502; Vol. 30, 1918, p. 221; Vol. 33, 1921, p. 174). Von Seidel's methods described in the German textbooks were intended for logarithmic computation. In the Logarithm Age, when all expressions used had to be in the form of products, and a formula with a sum or difference was anathema, artificial devices had to be resorted to, but the intro- duction of the calculating machine has altered that, and the ordinary formulæ of co-ordinate geometry can be used, though, of course, experience is required to ensure the most accurate arrangement. Tables are only needed for the sines of angles. A " T. I. M." calculating machine, and seven-figure natural sines, were employed for the work here described.

Owing to the time and labour involved in exact methods of studying the unit surfaces, various approximate methods have been in use.

Outline of Present Theoretical Knowledge.—For the chief facts stated in this outline the writer is also indebted to Mr. T. Smith. The conclusions " A " and " B " are from his article mentioned at the beginning of this Paper, and the paragraphs in quotation marks are from a note which he very kindly communicated on receiving the writer's results, in order to facilitate comparison. An earlier Paper may also be referred to (T. Smith, " Principles and Methods of Calculating Telescope Objectives," Transactions of the Optical Society, Vol. 18, 1917, p. 160 ; and Collected Researches of the National Physical Laboratory, Vol. 14, 1920, p. 99), where in particular it is pointed out that in thin systems the curvatures of the unit surfaces afford a measure of the amount of first order coma present in the image.

The unit (or principal) surfaces play a prominent part in the design of optical instruments. In elementary treatment based on a collinear, or point to point theory, they are taken as fixed planes, and their positions with respect to the foci give a general idea of the course of the rays, the position and size of the image, and the scope of the instrument. The collinear theory, however, in which small quantities above the first order are neglected, only holds for paraxial rays, and is insufficient when dealing with real instruments, especially photographic lenses where large

inclinations and apertures come into question. Consideration of a lens in the form of a solid sphere, in which any diameter may be taken as optic axis, will show the fallacy of the assumption that the principal surfaces are planes.

The next obvious step is to take into account some or all of the small quantities of the next order available—the so-called first order aberration terms. In some cases a good deal of information may thus be gained without taking the thickness into account, simply regarding all the refracting surfaces as cutting the axis in the same point, in which case the two axial unit points coincide with one another at this point.

For any such combination of thin lenses in air, it is known that with approximately unit magnification, images near the axis show no astigmatism, and are free from spherical aberration and coma. Consequently the image of a surface is here a single surface.

For thin lenses, when the rays are not greatly inclined to the axis :

A (1) The unit surfaces are in general spherical.

  (2) The curvatures depend on the lens shapes.

  (3) For a system of given power, the difference between the curvatures of the object unit surface and the image unit surface is a constant independent of the lens shapes.

  (4) The difference under (3) is of opposite sign to the difference between the incident and emergent wave-fronts as they pass the axial unit points.

  (5) The proper adjustment of the curvatures of the unit surfaces is essential to removal of aberrations.

The consideration of small quantities of the third order (first order aberrations) is insufficient to show whether the unit surfaces are fixed or not.

" For thick systems it is theoretically possible to obtain entirely free from all aberrations a plane image of a plane object for one particular magnification, and simultaneously abolish spherical aberration for the axial unit points. Then for beams with chief rays passing through these points the image will be free from all aberrations depending on skewness of rays, not only for one particular magnification, but for all magnifications. This is the most nearly perfect state of correction it is possible to achieve in a thick coaxial system for monochromatic light. It is not known how closely any real system can be constructed to agree with these properties, for in practice the surfaces must almost invariably be spherical, and the properties of stable transparent solids available as refracting media are exceedingly circumscribed. In addition, the necessity for fulfilling chromatic requirements, and considerations affecting the cost of manufacture, may necessitate deliberate departure from otherwise desirable conditions."

A thick system corrected in the theoretically most desirable manner has the following properties :—

B (1) There are no fixed unit surfaces, but a separate pair of surfaces for every object point in the aberrationless plane.

  (2) These unit surfaces are spherical.

(3) The centres of curvature of the unit surfaces are the aberrationless object and image points to which they correspond, and consequently

(4) The unit surfaces coincide with the wave-fronts as they pass through the unit points.

" The contradiction under (4) in the properties of the thin and ideal thick lenses is of particular interest, and accounts for some of the imperfections in results based on consideration of thin systems, imperfections which are most manifest when the aperture and field of view are large. Moreover as photographic lenses aim at reaching the conditions laid down for the ideal lens, but are composed of elements which are not far removed from thin lenses, it is instructive to see which type they approach more closely. It is assumed that the lenses are corrected for infinity, or for an object so distant that correction for infinity may be assumed. In this case, both in the thin and the ideal thick systems, the second unit surface would be made to coincide with emergent wave position, and any failure to reach the proper spherical form for these image-space surfaces will correspond to errors which the designer has consciously attempted to remove."

" From the point of view of the present investigation, particular interest attaches to the shape of the first unit surfaces. In a perfect lens they would be plane. In a thin lens their curvatures would be greater than those of their images, and of the same sign. The designers in all probability have made no conscious attempt to control their shapes. A study of the shapes of the first unit surfaces should be of value to lens designers."

Coming now to the present problem, it is well known that a small pencil of rays from an object point will in general after refraction no longer converge to a point ; a given ray will only meet two of the consecutive rays originally surrounding it. If the chief ray of the thin pencil lies in an axial plane, the chief ray will meet the consecutive rays in the meridional and sagittal planes, and will meet them in different points, the primary and secondary foci respectively. If we consider a family of chief rays obeying some law, for instance, rays passing through a fixed point, then as the object point describes a plane perpendicular to the axis, the primary and secondary foci will generally describe two different curved surfaces. In other words, the image of a plane perpendicular to the axis is not a plane, but a pair of curved surfaces. By suitably designing a lens this defect can be abolished for a particular position of object and image, or for a particular magnification, but not for all magnifications. In an ordinary photpgraphic lens the particular magnification selected for correction would naturally be that corresponding to the most usual position of the plate, and would be considerably less than unity. Hence for all other magnifications, including unity, curvature and astigmatism might be expected in the image.

The present Paper deals only with the secondary focus, consideration of the primary focus being deferred. The secondary focus is the simpler to deal with because it lies in the same axial plane as the object point, and, for a general ray, we may follow Mr. Smith in defining conjugate points as the points where a ray cuts an axial plane. The relation between object and image points is clearly a reciprocal one.

Taking the optic axis as axis of x, the first vertex as origin in the object

space, the last vertex as origin in the image space, and the direction the light travels as positive,

let u be the distance of object point P from first surface measured along the ray.

  v distance of secondary focus Q from last surface measured along the ray.

  x_0, y_0, z_0, co-ordinates of P.

  x_n, y_n, z_n co-ordinates of Q.

  X_1, Y_1, Z_1 co-ordinates of point of incidence.

  X_n, Y_n, Z_n co-ordinates of point of exit.

  L_0, M_0, N_0 direction-cosines of incident ray.

  L_n, M_n, N_n direction-cosines of refracted ray,

The expression of the fact that P and Q are conjugate gives one relation between u and v. If another relation is given, such as the magnification for small transverse lengths through Q and P (measured by the ratio of the distances of Q and P from the optic axis), then u and v can both be found. It is to be noted that the solution is unique.

Extremely neat and convenient formulæ for finding u and v in terms of the magnification m for a known ray have been given by Mr. T. Smith elsewhere (" On Tracing Rays through an Optical System," Proc. Phys. Soc., Vol. 27, 1915, p. 508), namely

$$u = U + \frac{\mu_0}{mK} \quad . \quad . \quad . \quad (1) ; \qquad v = V - \frac{\mu_n \cdot m}{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(the lettering is altered). The constants U, V are evidently the values of u, v for $m = \pm \infty$, $m = \pm 0$ respectively, and the corresponding points may be termed the principal oblique foci for the ray. K is a quantity which may be called the oblique power for the ray. If ξ, ξ' be the distances of any pair of conjugate points from these foci, then $\xi \xi' = -\frac{\mu_0 \mu_n}{K^2}$, just as on the axis.

U, V, K are calculated in the same way as the corresponding quantities in the analogous paraxial formulæ, except that, in the expression for the power k_λ of a single surface, $\mu_\lambda \cos \varphi'_\lambda - \mu_{\lambda-1} \cos \varphi_\lambda$ must be written instead of $\mu_\lambda - \mu_{\lambda-1}$, and the thickness t_λ must be measured obliquely along the ray instead of along the axis.

If the values u'', v'' of u, v for any magnification m'' have been found, the values for any other magnification m can be deduced in a moment from the formulæ (1, 2), since evidently

$$u - u'' = \frac{\mu_0}{K}\left(\frac{1}{m} - \frac{1}{m''}\right) \quad . \quad . \quad . \quad (3) ; \qquad v - v'' = -\frac{\mu_n}{K}(m - m'') . \quad . \quad . \quad . \quad (4)$$

A similar remark applies to the rectangular co-ordinates (x, y, z) (x'', y'', z'') of the conjugate points for the two magnifications. For the incident ray

$$x_0 - x_0'' = \frac{\mu_0 L_0}{K}\left(\frac{1}{m} - \frac{1}{m''}\right) ; \quad y_0 - y_0'' = \frac{\mu_0 M_0}{K}\left(\frac{1}{m} - \frac{1}{m''}\right) ; \quad \&c. \quad . \quad . \quad . \quad . \quad (5)$$

For the emergent ray

$$x_n - x_n'' = -\frac{\mu_n L_n}{K}(m - m'') ; \quad y_n - y_n'' = -\frac{\mu_n M_n}{K}(m - m'') \quad . \quad . \quad . \quad . \quad (6)$$

Suppose P, Q to be a pair of conjugate points for which there is accurate point-to-point representation, so that all rays from P are refracted through Q, and let m

be the corresponding magnification.   Then, if ρ be the distance of the 1st unit point from P, and σ the distance of the 2nd unit point from Q, on any such ray, (3) (4) give

$$\rho = \frac{\mu_0}{K}\left(\frac{1}{m}-1\right); \quad \sigma = -\frac{\mu}{K}(m-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

m is obviously constant for all rays through P and Q.   If K is also constant for all such rays, then the 1st unit point lies on a circle round P with radius $\frac{\mu_0}{K}\left(\frac{1}{m}-1\right)$, and the 2nd unit point lies on a circle round Q, with radius $-\frac{\mu n}{K}(m-1)$.   If P lies at infinite distance, then

$$\rho = \infty \; ; \; \sigma = -\frac{\mu_n}{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

It can be proved that K is constant for rays through any pair of object and image points on the aberrationless planes in the ideal thick lens.   Hence the results in " B " above.

After the rays required for the investigation had been traced by exact methods through the lens systems, the values of u and v corresponding to unit magnification were then calculated by the Smithian formulæ (1) (2), thus fixing the positions of the unit points.   For convenience in plotting, the rectangular co-ordinates of the unit points were then determined from these data.

The main investigation is confined to rays in an axial plane, since these afford an insight into the performance of the system, while involving much less time and labour than three-dimensional rays.   In the case of Cooke, however, a few rays in three dimensions are included, and are tabulated separately in Table VII.   They are incident at 20° to the axis in planes parallel to the plane ($z=0$) of the axial rays at

(I)   a point on rim of 1st surface 45° from bottom.
(II)   ,,   ,,   ,,   90°   ,,
(III) the point of intersection of the planes $y=0$ and $z=2/3$ radius of rim.

Also in the case of the Cooke some axial rays have been included which would ordinarily be stopped by mechanical means, and, as might be expected, the loci show extraordinary forms in these regions, for which the lens is not designed.   In the Tessar the case is different ; its sixth and seventh surfaces, forming the last convex lens, intersect at a distance of only 10·221 mm. from the axis, and obviously rays incident on the surfaces beyond this limit must not be considered, since even if the space between the produced surfaces were filled with glass the rays would strike them in the wrong order.   The ray for $\psi_0 = 5°$, full aperture, has however been included because it passes so near the limit, though just outside it, that any error caused thereby must be trifling.

The results of the investigation are given in the following tables.

It will be noticed that the paraxial unit points, or principal points, for the Tessar are in the middle of the lens, and only 1·0525 mm. apart, while for the Cooke they fall a little way behind the first surface, and are only 0.6755 mm. apart.   The points are not " crossed " in either case, the second principal point being on the image side of the first principal point.

While the main object of the investigation was to find the loci of the unit points

(or conjugate points for unit magnification) for oblique rays and large apertures, the positions of conjugate points for magnifications —0, — ∞, and (for the Tessar) —1, have also been calculated. These however are not given here, as the printing would be costly, and they can be computed from the data for the unit points by formulæ (5) (6) with the aid of the ray-constants which are tabulated below.

In the following tables, the axis of x is the optic axis, the origin being taken at the first vertex for the object, and at the last vertex for the image. The incident ray is defined by (1st) the inclination ψ_0 to the optic axis, and (2nd) the aperture. For the present purpose aperture is measured by the length of the perpendicular on the ray from the first vertex, the description "full aperture" signifying that this perpendicular is equal to the semi-diameter of the front lens.

TESSAR PHOTOGRAPHIC LENS, 1907.

$$f_D = 100. \qquad \text{Aperture ratio} = \frac{f}{4\cdot5}$$

				Radius.	Thickness.	μ
1	...	...	...	+ 26·3		
					3·7	1·61342
2	...	...	...	∞		
					4·0	1
3	...	...	...	— 58·1		
					1·7	1·57391
4	...	...	...	+ 23·9		
					5·3	1
5	...	...	...	—146·7		
					1·7	1·53
6	...	...	...	+ 22·3		
					4·6	1·61451
7	...	...	...	— 36·3		
					Total 21·0	

TABLE I.—*Co-ordinates of Conjugate Points for Unit Magnification.*
(Unit Points.)

Ray.		Object.		Image.	
ψ_0	Aperture.	x_0	y_0	x_n	y_n
0°	0	+8·79608	0	—11·15139	0
	$+\frac{1}{3}$	8·80748	+3·703	—11·08460	+ 3·70362
	$+\frac{2}{3}$	8·85050	7·407	—10·87075	+ 7·40738
	+Full	8·95401	11·1	—10·46590	+11·1
5°	—Full	9·87600	—10·28952	— 9·63045	—10·28952
	$-\frac{2}{3}$	9·47512	— 6·60674	—10·31733	— 6·60672
	$-\frac{1}{3}$	9·10210	— 2·92152	—10·83964	— 2·92137
	0	8·75133	+ 0·76565	—11·22177	+ 0·76565
	$+\frac{1}{3}$	8·42827	+ 4·45523	—11·46357	+ 4·45523
	$+\frac{2}{3}$	8·16195	+ 8·14978	—11·53380	+ 8·14979
	+Full	8·05315	+11·85811	—11·32532	+11·85809
20°	—Full	12·29516	— 7·34912	— 7·86657	— 7·34909
	$-\frac{2}{3}$	10·88695	— 3·92027	— 9·54039	— 3·92028
	$-\frac{1}{3}$	9·53630	— 0·47047	—10·98399	— 0·47046
	0	8·29120	+ 3·01776	—12·19216	+ 3·01775
	$+\frac{1}{3}$	7·36882	+ 6·62343	—12·99609	+ 6·62346

TABLE II.—*Ray Constants.*

Oblique power.	Object.		Image.	
$\dfrac{1}{K}$	$\dfrac{\text{Cos } \psi_0}{K}$	$\dfrac{\text{Sin } \psi_0}{K}$	$\dfrac{\text{Cos } \psi_n}{K}$	$\dfrac{\text{Sin } \psi_n}{K}$
99·80777	+99·80777	0	+99·80777	0
99·61340	99·6134	0	99·5445	− 3·70376
99·2357	99·2357	0	98·9589	− 7·40739
99·6087	99·6087	0	98·9870	−11·1111
99·8553	99·4753	+ 8·70295	98·0324	+18·9925
99·6036	99·2246	8·68102	98·4234	+15·2877
99·9713	99·5909	8·71307	99·2919	+11·6349
100·1162	99·7352	8·72570	99·7992	+ 7·96007
99·8629	99·4829	8·70362	99·7725	+ 4·24841
99·4283	99·0499	8·66574	99·4269	+ 0·51591
99·7304	99·3509	8·69208	99·6802	− 3·16600
105·6271	99·2570	36·1266	96·2649	+43·4757
105·5281	99·1640	36·0927	97·6480	+40·0130
105·7371	99·3604	36·1642	99·1879	+36·6347
105·4313	99·0730	36·0596	100·1199	+33·0418
96·5888	97·5359	35·5002	99·6978	+28·8767

TABLE III.—*Curvature of Unit Surfaces of Tessar Lens.*

Radii and Centres of Circles through Unit Points on Certain Rays (ψ_0=Inclination to Axis of Incident Ray).

Rays.				Object.			Image.		
ψ_0	Apertures.			Radius.	Co-ordinates of centre.		Radius.	Co-ordinates of centre.	
					x	y		x	y
0°	$-\frac{1}{3}$	0	$+\frac{1}{3}$	601·65	610·44	0	102·72	91·573	0
	$-\frac{2}{3}$	0	$+\frac{1}{3}$	504·16	512·95	0	97·898	86·747	0
	−Full	0	+Full	390·94	399·73	0	90·392	79·241	0
5°	$-\frac{1}{3}$	0	$+\frac{1}{3}$	492·93	499·63	45·605	98·012	86·302	9·0098
	$-\frac{2}{3}$	0	$+\frac{2}{3}$	406·71	413·86	36·816	92·827	81·293	8·3737
	−Full	0	+Full	288·49	296·27	24·431	83·362	71·900	7·0883
20°	$-\frac{2}{3}$	$-\frac{1}{3}$	0	139·25	123·95	42·336	59·089	44·220	20·603*
	−Full	$-\frac{1}{3}$	$+\frac{1}{3}$	87·36	91·92	28·580	50·087	36·065	16·707

* A circle with same centre, but radius 58·90, would pass through unit point on ray at $+\frac{1}{3}$ Ap., and a circle with same centre, but radius 59·11, would pass through unit point on ray at −Full Ap.

COOKE PHOTOGRAPHIC LENS, 1894.

$f_D = 100$.   Aperture ratio $= f/7\cdot7$.

				r	t	μ
1	...	...	...	$+14\cdot6$		
					$2\cdot99$	$1\cdot6114$
2	...	...	...	$-101\cdot3$		
					$0\cdot38$ air	1
3	...	...	...	$-55\cdot9$		
					$0\cdot46$	$1\cdot5482$
4	...	...	...	$+13\cdot3$		
					$8\cdot95$ air	1
5	...	...	...	$+1012\cdot0$		
					$1\cdot83$	$1\cdot6114$
6	...	...	...	$-69\cdot8$		
			Total		$14\cdot61$	

TABLE IV.—*Co-ordinates of Unit Points.*

Ray.		Object.		Image.	
ψ_0	Ap.	x_0	y_0	x_n	y_n
0°	0	$+1\cdot318334$	0	$-12\cdot61616$	0
	$+\frac{2}{3}$	$1\cdot34903$	$+4\cdot329004$	$-12\cdot5070$	$+4\cdot32885$
	$+$Full	$1\cdot43258$	$6\cdot493506$	$-12\cdot3296$	$+6\cdot49350$
20°	$-$Full	$3\cdot67797$	$-5\cdot57158$	$-10\cdot33238$	$-5\cdot57155$
	$-\frac{2}{3}$	$2\cdot87204$	$-3\cdot56149$	$-11\cdot2484$	$-3\cdot56146$
	0	$1\cdot29869$	$+0\cdot47268$	$-12\cdot8903$	$+0\cdot47274$
	$+\frac{2}{3}$	$-0\cdot83603$	$+4\cdot30254$	$-14\cdot7926$	$+4\cdot30246$
	$+$Full	$-2\cdot522$	$+5\cdot99$	$-15\cdot94$	$+6\cdot046$
30°	$-$Full	$+4\cdot90668$	$-4\cdot66518$	$-\ 9\cdot52814$	$-4\cdot66521$
	$-\frac{2}{3}$	$3\cdot84048$	$-2\cdot78140$	$-10\cdot7561$	$-2\cdot78143$
	0	$1\cdot39853$	$+0\cdot8072$	$-13\cdot335$	$+0\cdot8082$

TABLE V.—*Ray Constants.*

$\dfrac{1}{K}$	Object.		Image.		
	$\dfrac{\cos \psi_0}{K}$	$\dfrac{\sin \psi_n}{K}$	$\dfrac{\cos \psi_n}{K}$	$\dfrac{\sin \psi_n}{K}$	
$100\cdot36558$	$100\cdot36558$	0	$100\cdot3656$	0	
$100\cdot26068$	$100\cdot2607$	0	$100\cdot1672$	$4\cdot3289$	
$100\cdot8236$	$100\cdot8236$	0	$100\cdot6143$	$6\cdot4935$	
$108\cdot0561$	$101\cdot5395$	$36\cdot9574$	$99\cdot3349$	$42\cdot5288$	
$105\cdot7992$	$99\cdot4187$	$36\cdot1854$	$98\cdot0492$	$39\cdot7470$	
$106\cdot3902$	$99\cdot9741$	$36\cdot3876$	$100\cdot1450$	$35\cdot9149$	
$109\cdot4811$	$102\cdot8785$	$37\cdot4447$	$104\cdot3464$	$33\cdot1422$	Stopped
$113\cdot39$	$106\cdot55$	$38\cdot78$	$108\cdot55$	$32\cdot76$	Stopped
$119\cdot1260$	$103\cdot1661$	$59\cdot5630$	$100\cdot3292$	$64\cdot2281$	
$115\cdot1086$	$99\cdot6870$	$57\cdot5543$	$98\cdot0285$	$60\cdot3357$	
$120\cdot4529$	$104\cdot316$	$60\cdot2269$	$104\cdot7781$	$59\cdot4187$	Stopped

<div align="center">

TABLE VI.—*Curvature of Unit Surfaces of Cooke Lens.*

Radii and Centres of Circles through Unit Points on Certain Rays.

</div>

Rays.		Object.			Image.		
ψ_0	Apertures.	Radius.	Co-ordinates of centre.		Radius.	Co-ordinates of centre.	
0°	$-\frac{2}{3}$ 0 $+\frac{2}{3}$	305·23	306·55	0*	85·874	73·258	0
	$-$Full 0 $+$Full	184·59	185·91	0	73·709	61·092	0
	$-$Full $-\frac{2}{3}$ 0	103·14	104·45	$+1\cdot4333$	57·489	44·868	0·7161
20°	$-$Full $-\frac{2}{3}$ 0	346·2	322·0	123·2	79·92	61·93	28·57

* The unit points on the rays through full aperture lie only 0·104 mm. within this circle.

<div align="center">

COOKE LENS.—RAYS NOT IN AN AXIAL PLANE.

$L_0 = \cos 20° = 0\cdot9396926$, $M_0 = \sin 20° = 0\cdot3420201$, $N_0 = 0°$.

</div>

Ray.	Co-ordinates of Point of Incidence.		
I. at 45° Pt.	$+1\cdot5235183$	$-4\cdot5916025$	$+4\cdot5916025$
II. ,, 90° ,,	$+1\cdot5235183$	0	$+6\cdot4935065$
III. ,, Pt. $\frac{2}{3}$ horizontally	$+0\cdot6565528$	0	$+4\cdot329004$

<div align="center">

TABLE VII.—*Co-ordinates of Unit Points.*

</div>

Ray.	Object.			Image.		
	x_0	y_0	z_0	x_n	y_n	z_n
I.	$+3\cdot08357$	$-4\cdot02379$	$+4\cdot59160$	$-10\cdot90850$	$-4\cdot02379$	$+4\cdot59160$
II.	$+1\cdot33969$	$-0\cdot0669$	$+6\cdot49351$	$-12\cdot60245$	$-0\cdot06631$	$+6\cdot49345$
III.	$+1\cdot30300$	$+0\cdot23529$	$+4\cdot329004$	$-12\cdot78241$	$+0\cdot23527$	$+4\cdot32900$

<div align="center">

TABLE VIII.—*Ray Constants.*

</div>

Ray	$\frac{1}{K}$	Object.			Image.		
		$\frac{L_0}{K}$	$\frac{M_0}{K}$	$\frac{N_0}{K}$	$\frac{L_n}{K}$	$\frac{M_n}{K}$	$\frac{N_n}{K}$
I.	108·1397	101·6180	36·9859	0	99·9569	41·0097	$-4\cdot59160$
II.	108·4210	101·8824	37·0822	0	101·6509	37·1488	$-6\cdot49345$
III.	106·8006	100·3598	36·5280	0	100·3529	36·2927	$-4\cdot32900$

The co-ordinates of conjugate points for any magnification m can quickly be calculated from the tabulated co-ordinates of the unit points by (5) (6). For rays parallel to $z=0$ we have for the incident side

$$x = x_0 - \frac{\cos \psi_0}{K}\left(1 - \frac{1}{m}\right) \qquad\qquad y = y_0 - \frac{\sin \psi_0}{K}\left(1 - \frac{1}{m}\right)$$

and for the emergent side

$$x = x_n + \frac{\cos \psi_n}{K}(1 - m) \qquad\qquad y = y_n + \frac{\sin \psi_n}{K}(1 - m)$$

Example.

The figures are for rays incident at 20° to the axis at—full aperture.

TABLE IX.—*Co-ordinates of other Conjugate Points.*

Ray $\psi_0 = 20°$. Ap.—Full.

Magnification.	Object.		Image.	
	x	y	x	y
−0	− ∞	− ∞	$x_n + \dfrac{\cos \psi_n}{K}$ (Second Prin. Oblique Focus)	$y_n + \dfrac{\sin \psi_n}{K}$
Tessar	− ∞	− ∞	+88·3984	+36·1266
Cooke	− ∞	− ∞	+89·0026	+36·9573
−∞	$x_0 - \dfrac{\cos \psi_0}{K}$ (First Prin. Oblique Focus)	$y_0 - \dfrac{\sin \psi_0}{K}$	+ ∞	+ ∞
Tessar	−86·9618	−43·4757	+ ∞	+ ∞
Cooke	−97·8616	−42·5289	+ ∞	+ ∞
−1 (Inverted equal image)	$x_0 - \dfrac{2 \cos \psi_0}{K}$	$y_0 - \dfrac{2 \sin \psi_0}{K}$	$x_n + \dfrac{2 \cos \psi_n}{K}$	$y_n + \dfrac{2 \sin \psi_n}{K}$
Tessar	−186·219	−79·6023	+184·663	+79·6024
Cooke	−199·401	−79·4864	+188·337	+79·4861

As a matter of fact the positions of the oblique principal foci are found in the process of determining the unit points.

CONCLUSIONS.

(As stated above, only secondary or transverse foci are here considered.)

General.

It is at once evident from the results that for chief rays in an axial plane the unit points on either side lie scattered over a considerable area, and hence for general rays in space the loci are three-dimensional. The loci of the unit points will only be surfaces when the chief rays are bound by some condition, such as passing through a fixed point.

Consider, for example, rays from a given object point at infinity, or parallel rays of given inclination. The surface locus is represented in the plane of the paper by a line section. The results show that

C (1) The unit surfaces are not fixed, but shift with the inclination of the parallel rays, *i.e.*, withthe position of the infinitely distant object point.

(2) The unit surfaces are not spherical, the curvature of the plane section changes with aperture.

(3) The curvatures alter also with the inclination of the rays, *i.e.*, with the distance from the axis of the infinitely distant object point.

(4). In all cases examined, within the region for which the lens is designed, the curvature of both first and second unit loci for parallel chief rays is positive (convex to the light source), and the second locus is more curved than the first.

(5) The unit surfaces not being plane or spherical cannot coincide with wave-fronts.

Comparing these results with " A " and " B " in the " outline " above, on the assumption that both lenses are corrected for infinity, or, at least, for a very distant object, it is seen that the facts do not coincide with the theoretical conclusions either for a thin or an ideal thick system, but that they accord more nearly with the latter, especially as regards (4) the sign of the difference of curvatures of the two unit surfaces.

If instead of a pencil of rays from a given object point, we take rays satisfying some other condition, such as passing through the vertex of the lens, or having a given aperture, *i.e.*, having H, the perpendicular on the ray from the vertex, given, then quite different forms are obtained for the unit surfaces.

So much for general deductions. Some individual characteristics of the lens may now be considered.

Tessar.

For parallel rays, the curvature of the unit surfaces, which, as might be expected, is very slight for zero inclination near the axis, appears to increase fairly regularly with inclination, and with aperture.

For rays through the vertex of the lens, or aperture=0, the plane section of the unit locus is a rather small arc curving rapidly to the object side with increasing inclination, the curvature being more marked in the second locus. With increasing apertures the arcs lengthen. For aperture −2/3, the arcs slope down to the right on the diagrams ; for −full aperture the unit points for inclinations 0° and 20° are nearly level. The aggregate second locus for rays in the plane of the paper appears to narrow almost to a point on the optic axis in the region of the unit point for paraxial rays, and then to widen rapidly as the distance from the axis increases. The first locus, however, appears to have a narrow neck about 1·5 mm. above the axis, near where the unit point for inclination 10° would lie, and to widen rapidly above and below this neck.

Cooke.

For parallel rays the 20° locus appears to be flatter than the corresponding portion of the 0° locus, but it is difficult to compare pencils of widely different inclination, where rays of the same aperture play different roles. For rays through the vertex, of inclinations 0° to 30°, the diagram shows that the first unit surface, or, rather, its section, curves only slightly, and to the right or image side, while the second surface curves quickly to the left. The loci for apertures −2/3, and −full, slope up to the right or image side.

The writer's best thanks are due to Mr. T. Smith for encouragement and facilities granted when starting the work, and for valuable criticism and information when discussing the completed results.

VIII. *Vibration Galvanometers with Asymmetric Moving Systems.* By R. Ll. Jones.

ABSTRACT.

In the theory of the vibration galvanometer with one degree of freedom certain specified conditions are assumed to hold. If these conditions are not satisfied in a galvanometer, the instrument will show multiple resonance. This indicates the need of a more general theory. The theory of vibrations of a system with two degrees of freedom is briefly given, expressions for the amplitudes of the forced vibrations are deduced, and the conditions for resonance ascertained. The results are applied to a galvanometer in which the moving system is asymmetrically hung on a laterally yielding axis, and it is shown that the formula for the amplitude is capable of reproducing with fair accuracy the sensitivity curve of the galvanometer, which shows multiple resonance. Asymmetry always lowers the sensitivity of the resonance, and the loss can be ascertained by the help of the sensitivity curve. The analogy to the vibrations in coupled circuits is pointed out.

I wish to express my gratitude to A. Campbell for helpful criticism, advice and references.

CONTENTS.

I. Theory of vibration galvanometer with one degree of freedom.
II. Vibration galvanometer showing multiple resonance.
III. Resonance in vibrating systems with two degrees of freedom.
IV. Vibrations of Asymmetric systems.
       (*a*) The equations of motion.
       (*b*) The forced vibrations.
       (*c*) The free vibrations.
       (*d*) Application.
V. Effect of asymmetry on sensitivity.
IV. Mechanical analogue of coupled circuits.
VII. Conclusion ; Summary.

I. Theory of the Vibration Galvanometer with One Degree of Freedom.

The theory of the vibration galvanometer has been considered by Wenner,[*] A. Campbell,[†] Butterworth,[‡] and others.[§] In most of these investigations the moving system is supposed to have only one degree of freedom—namely, freedom to rotate about the axis of suspension and control. The theory so developed is applicable to galvanometers in which

       (*a*) The axis is rigid and does not yield to forces tending to produce lateral displacements ;

       (*b*) The axis, although not rigid, passes through the centre of mass of the moving system and is a principal axis at that point.

For, in the first case, any reactions on the axis called into play by the angular motion of the system can produce no displacement ; while, in the second case, no reactions

* Wenner ; Bulletin Bureau of Standards, Vol. 6, pp. 347 (1910).
† A. Campbell ; Proc. Phys. Soc., Vol. 26, Part 2, pp. 120 (1914).
‡ Butterworth ; Proc. Phys. Soc., Vol. 24, Part 2, pp. 75 (1912).
§ Full references will be found in Dictionary of Applied Physics, Vol. 2, pp. 979-980.

arise and no constraint is necessary in order to secure that the displacement of the system shall continue to be one of angular rotation alone round the axis, provided that the external forces which may be impressed on it reduce to a couple round this axis. The impressed forces will reduce to a couple if the field and the moving system have this axis as an axis of symmetry.

In a vibration galvanometer the axis is usually not rigid, and it may happen through errors of construction that it does not pass through the centre of mass of the moving system. The system may be mechanically and electrically unsymmetrical with regard to the axis, and the field may also be unsymmetrical. In these circumstances the motion of the system will not be restricted to simple angular displacements round the axis and a more general theory is required.

II. Vibration Galvanometer Showing Multiple Resonance.

It is well known that if the free period of a vibration galvanometer with only one degree of freedom be kept constant while the frequency of the current passing through it is changed, the galvanometer shows high sensitivity at and near the resonating frequency. At all other frequencies differing by more than a few per cent.
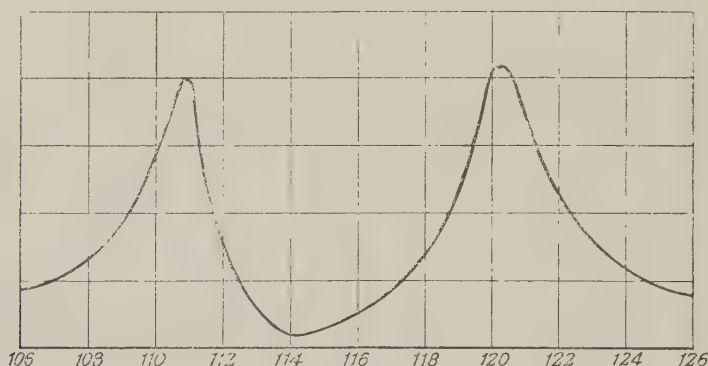
FIG. 1.

from the resonating frequency the sensitivity is small. Rosa and Grover,[*] however, found that a vibration galvanometer may show high sensitivity at two different frequencies. For instance, one of the galvanometers they used gave a maximum sensitivity at a frequency of 110·6 and also at 120 periods per second, while at 114 the sensitivity was a minimum. The diagram in Fig. 1 shows the sensitivity of their galvanometer over the range 106 to 126 per second. They draw special attention to one remarkable feature of this diagram—namely, that at frequencies between 110·6 and 120 the sensitivity is much lower than it is immediately outside these values. The ordinary theory is incapable of accounting for these observations.

The moving system of this galvanometer was a magnet. It was attached to a wire clamped at two points, and the control could be varied by altering the distance between these points and the tension of the wire. Wenner[†] states that this axis

---

[*] Rosa and Grover; Bulletin Bureau of Standards, No. 3, p. 298 (1905).
[†] *Loc. cit.*

did not pass through the centre of mass of the moving system, and that this lack of mechanical symmetry was introduced on the repair of a broken suspension. The system had evidently more than one degree of freedom, owing to the lateral yielding of the wire during the motion of the unbalanced system round it, and in attempting to account for its motion and mechanical behaviour we will apply the theory of the vibrations of a system with two degrees of freedom. A brief outline of the theory and some deductions from it applicable to the vibration galvanometer, more especially the conditions for resonance, are given in § III. below.* It will be shown that the theory gives a formula for the amplitude of the forced vibration which is capable of reproducing with considerable accuracy the curve of sensitivity of Rosa and Grover's galvanometer over the range of frequencies given in Fig. 1, and enables us to deduce from the sensitivity curve the mechanical caracteristics of the moving system.

III. Resonance in Vibrating Systems with Two Degrees of Freedom.

Consider a system with two degrees of freedom making small vibrations about a position of stable equilibrium. Let its position at any instant be defined by two independent co-ordinates q_1, q_2, and let $q_1 = 0$, $q_2 = 0$ in the position of equilibrium. Then if q_1 and q_2 are suitably chosen, we can express the kinetic energy T, the dissipation function F, and the potential energy V of the system in the forms

$$
\left.
\begin{aligned}
T &= \tfrac{1}{2}(a_1\dot{q}_1{}^2 \times 2a_{12}\dot{q}_1\dot{q}_2 + a_2\dot{q}_2{}^2) \\
F &= \tfrac{1}{2}(b_1\dot{q}_1{}^2 \qquad\qquad + b_2\dot{q}_2{}^2) \\
V &= \tfrac{1}{2}(c_1 q_1{}^2 \qquad\qquad + c_2 q_2{}^2)
\end{aligned}
\right\} \qquad \ldots \ldots \ldots \quad (1)
$$

in which $a_1 \ldots b_1 \ldots \ldots \ldots c_2$ are constants, as we are considering small oscillations.

By Lagrange's method, the equations of motion of the system are

$$
\left.
\begin{aligned}
a_1\ddot{q}_1 + b_1\dot{q}_1 + c_1 q_1 + a_{12}\ddot{q}_2 &= Q_1 \\
a_{12}\ddot{q}_1 + a_2\ddot{q}_2 + b_2\dot{q}_2 + c_2 q_2 &= Q_2
\end{aligned}
\right\} \qquad \ldots \ldots \ldots \ldots \quad 2)
$$

where $Q_1 Q_2$ are the applied forces of corresponding types.

Let $Q_1 = e^{ipt}$ and $Q_2 = 0$ in (2), where $i = \sqrt{-1}$. Writing D for $\dfrac{d}{dt}$ we get

$$
\begin{aligned}
q_1 &= \left[\frac{a_2 D^2 + b_2 D + c_2}{(a_1 D^2 + b_1 D + c_1)(a_2 D^2 + b_2 D + c_2) - a_{12}{}^2 D^4}\right] e^{ipt} \\[2mm]
&= \frac{(c_2 - a_2 p^2) + i \cdot b_2 p}{\{(c_1 - a_1 p^2) + i \cdot b_1 p\}\{(c_2 - a_2 p^2) + i \cdot b_2 p\} - a_{12}{}^2 \cdot p^2} \cdot e^{ipt} \\[2mm]
&= \frac{S_2}{R}\, e^{i(pt - \chi + \psi_2)}
\end{aligned}
$$

$$
\left.
\begin{aligned}
\\
\\
\end{aligned}
\right\} \qquad \ldots \ldots \ldots \ldots \ldots \quad (3)
$$

and

$$
q_2 = \frac{a_{12} \cdot p^2}{R}\, e^{i(pt - \chi)}
$$

---

* A fuller treatment of the theory is given in Rayleigh's Theory of Sound, Vol. 1, Chap. 4.

Similarly, when $Q_1 = 0$, $Q_2 = e^{ipt}$, we have

$$\left.\begin{aligned}
q_1 &= \frac{a_{12} \cdot p_2}{R} e^{i(pt-\chi)} \\
q_2 &= \frac{S_1}{R} e^{i(pt-\chi+\psi_1)}
\end{aligned}\right\} \qquad \cdots \cdots \cdots \quad (4)$$

In (3) and (4)

$$\left.\begin{aligned}
S_1^2 &= a_1^2 p^4 \left[x_1^2 + \frac{\lambda_1^2}{p^2} \right] \\
S_2^2 &= a_2^2 p^4 \left[x_2^2 + \frac{\lambda_2^2}{p^2} \right] \\
R^2 &= a_1^2 a_2^2 p^8 \left[\left(x_1 x_2 - \delta - \frac{\lambda_1 \lambda_2}{p^2} \right)^2 + \left(\frac{\lambda_2 x_1 + \lambda_1 x_2}{p} \right)^2 \right] \\
\tan \psi_1 &= \lambda_1 / p x_1 \\
\tan \psi_2 &= \lambda_2 / p x_2 \\
\tan \chi &= \left(\frac{\lambda_2 x_1 + \lambda_1 x_2}{p} \right) \Big/ \left(x_1 x_2 - \delta - \frac{\lambda_1 \lambda_2}{p} \right)
\end{aligned}\right\} \qquad \cdots \cdots \quad (5)$$

where

$$\left.\begin{aligned}
X_1 &= \frac{\omega_1^2 - p^2}{p^2} & X_2 &= \frac{\omega_2^2 - p^2}{p^2} \\
\omega_1^2 &= \frac{c_1}{a_1} & \omega_2^2 &= \frac{c_2}{a_2} \\
\lambda_1 &= \frac{b_1}{a_1} & \lambda_2 &= \frac{b_2}{a_2} \\
& & \delta &= \frac{a_{12}^2}{a_1 a_2}
\end{aligned}\right\} \qquad \cdots \cdots \quad (6)$$

The real parts of (3) give the vibrations in q_1 and q_2 when $Q_1 = \cos pt$ and $Q_2 = 0$. When $Q_1 = 0$ and $Q_2 = \cos pt$ the vibrations in q_1 and q_2 are given by the real parts of (4).

From these it follows that when $Q_1 = \cos pt$ and $Q_2 = r \cos (pt + \varphi)$ the vibration in q_2 is given by

$$q_2 = \frac{1}{R} [r S_1 \cos (pt + \varphi - \chi + \psi_1) + a_{12} p^2 \cos (pt - \chi)] \quad \cdots \cdots \quad (7)$$

This vanishes provided

$$\left.\begin{aligned}
\varphi + \psi_1 &= \pi \\
r &= a_{12} \cdot p^2 / S_1
\end{aligned}\right\} \qquad \cdots \cdots \cdots \quad (8)$$

From (3) . . . (8) we get the following :—

　　(i) If $a_{12} = 0$, the forced oscillation in q_2 due to Q_1 given in (3) vanishes. In this case the co-ordinates q_1, q_2 are normal co-ordinates, and the oscillation in q_1

is the same as if the system had only one degree of freedom. Since $\delta=0$ when $a_{12}=0$, R will be given by

$$R^2 = a_1{}^2 a_2{}^2 p^8 \left[\left(X_1{}^2 + \frac{\lambda_1{}^2}{p^2} \right) \left(X_2{}^2 + \frac{\lambda_2{}^2}{p^2} \right) \right]$$

and
$$\frac{S_2}{R} = \frac{1}{a_1 \{(\omega_1{}^2 - p^2)^2 + \lambda_1{}^2 p^2\}^{\frac{1}{2}}} \quad \dots \dots \dots \dots \quad (9)$$

which is identical with the expression for the amplitude in a system with only one degree of freedom.

(ii) The values of ω_2 and p being given, let us determine what value ω_1 must have in order that S_2/R in (3) may be a maximum. The amplitude S_2/R will be a maximum when

$$\omega_1{}^2 = p^2 \left[1 + \delta X_2 \bigg/ \left(X_2{}^2 + \frac{\lambda_2{}^2}{p^2} \right) \right] \quad \dots \dots \dots \quad (10)$$

When $\delta=0$ relation (10) reduce to

$$\omega_1 = p$$

which is the condition for resonance in a system with one degree of freedom when resonance is obtained by tuning the system.

When ω_1 has the value given in (10)

$$S_2/R = \frac{1}{a_1 p} \left(X_2{}^2 + \frac{\lambda_2{}^2}{p^2} \right) \bigg/ \left[\lambda_1 \left(X_1{}^2 + \frac{\lambda_1{}^2}{p^2} \right) + \lambda_2 \delta \right] \quad \dots \dots \quad (11)$$

When $\delta=0$ relation (11) reduces to

$$S_2/R = \frac{1}{a_1 \lambda_1 p} \quad \dots \dots \dots \dots \dots \dots \quad (12)$$

which is that (9) reduces to when $\omega_1=p$ and gives the amplitude at resonance in a system with one degree of freedom.

Denoting the amplitude in (11) by $A\delta$ and that in (12) by A_0, we have

$$A\delta : A_0 = \left(X_2{}^2 + \frac{\lambda_2{}^2}{p^2} \right) : \left(X_2{}^2 + \frac{\lambda_2{}^2}{p^2} + \frac{\lambda_2}{\lambda_1} \delta \right) \quad \dots \dots \quad (13)$$

As all the quantities in this ratio are positive, it will be seen that the presence of a term involving the product $(q_1 q_2)$ in the expression for the kinetic energy means that the amplitude at resonance is always less than it would be were $a_{12}=0$, *provided other conditions remain unchanged.* A fuller discussion is given in § V. below.

(iii) The values ω_1 and p being given, S_2/R the amplitude of q_1 in (3) will be a maximum or minimum when ω_2 has a value which satisfies the equation.

$$X_2{}^2 - \frac{1}{X_1} \left(\delta + \frac{2\lambda_1 \lambda_2}{p^2} \right) X_2 - \frac{\lambda_2{}^2}{p^2} = 0 \quad \dots \dots \dots \quad (14)$$

When $\lambda_1=\lambda_2=0$ this equation reduces to

$$\left. \begin{array}{c} X_2 = 0 \\ X_1 X_2 = \delta \end{array} \right\} \quad \dots \dots \dots \dots \dots \quad (15)$$

or

These conditions are discussed in the following sub-section (iv) :—

(iv) When ω_1 and ω_2 are fixed the values of p which make the amplitude of q_1 given in (3) a maximum or a minimum are readily obtained if the damping be

neglected. The values so obtained will not differ materially from the values making S_2/R a maximum or a minimum when the damping is taken into account, unless the damping be excessive.* Thus the amplitude will be

a minimum when $\qquad$ $X_2=0$ or $p=\omega_2$

a maximum when $\qquad\qquad\qquad\quad X_1X_2=\delta$ $\left.\begin{array}{c}\\\\\end{array}\right\}$ . . . . (16)

Hence two values of p make the amplitude a maximum, one value being greater and the other smaller than either ω_1 and ω_2. When p has either of these values the amplitude, which will be denoted by A, is given very approximately by

$$A=\frac{X_2}{a_1p(\lambda_2X_1+\lambda_1X_2)} \qquad . . . . (17)$$

Let $p_1\,p_2$ denote the values of p which satisfy the relation $X_1X_2=\delta$ in (16). Let $p_1>\omega_1>\omega_2>p_2$ and suppose that ω_1 and ω_2 are far apart. Then when $p=p_1$ X_2 will be numerically much greater than X_1 at this frequency and the amplitude A_1 given by (17) becomes

$$A_1=\frac{1}{a_1\lambda_1p_1} \qquad . . . . (18)$$

which is identical with (12). Hence in such a case the amplitude at resonance is nearly the same as if the system had only one degree of freedom. If A_2 denotes the amplitude at the frequency corresponding to $p=p_2$, then, in general, $A_2<A_1$. When $p_2>\omega_2>\omega_1>p_1$, A_2 is always less than A_1.

(v) The amplitude of q_2 in (3) is a maximum when

$$\omega_1{}^2=p^2\left[\,1+\delta X_2/\left(X_2{}^2+\frac{\lambda_2{}^2}{p^2}\right)\right] \qquad . . . . (19)$$

ω_2 and p being kept unchanged, and also when

$$\omega_2{}^2=p^2\left[\,1+\delta X_1/\left(X_1{}^2+\frac{\lambda_1{}^2}{p^2}\right)\right]. \qquad . . . . (20)$$

ω_1 and p being kept unchanged.

(vi) When $Q_1=\cos pt$ and a constraint is imposed so that $q_2=0$, the magnitude and phase of the force exerted by the constraint are given by (8), and $\dfrac{a_{12}\cdot p^2}{S_1}\cos(pt-\psi_1)$ gives the reaction. The reaction vanishes when $a_{12}=0$. The system in this case vibrates under Q_1 as if it had only one degree of freedom and no constraint is necessary in order to secure that $q_2=0$ for all values of t (§ I. b).

When a constraint is imposed so that $q_2=0$, the amplitude of q_1 is given by (9) when $Q_1=\cos pt$.

In the following section the results given above are applied to the vibration galvanometer with an asymmetrically mounted moving system. The moving system may be a magnet forced to vibrate by periodic variations in the field produced by the current or a coil through which the alternating current passes suspended in a constant field.

---

* In a vibration galvanometer the damping should not be large, in order that the instrument may have the necessary selectivity and sensitivity.

IV. Vibrations of Asymmetric Systems.

(a) Equations of Motion.

Let nS (Fig. 2) be the magnet attached to the middle point O of the wire AB. The wire is fixed at A and B and is at right angles to the direction Oy of the uniform magnetic field H. Take Ox at right angles to Oy and OA as axis of x. In the equilibrium position ns is at right angles to H. Let G be the centre of mass of the moving system, and let its co-ordinates in the position of rest be $(a, 0)$. We will suppose that the system is symmetrical with respect to the plane xy. During the motion, O may be displaced. Let $E(xy)$ be its position at a time t and θ the angular displacement of ns round AB. Let m be the mass of the moving system and mk^2 its moment of inertia about an axis through G parallel to AB. Let m^1 be the effective mass* of the wire AB. Since $OG=a$, we have for the kinetic energy T of the moving parts

$$T=\tfrac{1}{2}(m+m^1)(\dot{x}^2+\dot{y}^2)+\tfrac{1}{2}m(a^2+k^2)\dot{\theta}^2-ma \sin \theta . \dot{x}\dot{\theta}+ma \cos \theta . \dot{y}\dot{\theta}.$$

The only co-ordinate explicitly contained in T is θ. As we are considering small oscillations about the position $x=0$, $y=0$, $\theta=0$, we may give to θ where it occurs in T its equilibrium value $\theta=0$. Further, we shall suppose that m^1 is small compared with m and shall write m for $(m+m^1)$ in the coefficient of $(\dot{x}^2+\dot{y}^2)$. Thus we get

$$T=\tfrac{1}{2}m[\dot{x}^2+\dot{y}^2+(a^2+k^2)\dot{\theta}^2+2a\dot{\theta}\dot{y}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This is also the expression for the kinetic energy of the moving coil system a^1b^1 shown in Fig. 3. The potential energy V is given by

$$V=\tfrac{1}{2}v(x^2+y^2)+\tfrac{1}{2}\mu\theta^2$$

Let
$$F=\tfrac{1}{2}(k_x . x^2+k_y . y^2+k . \dot{\theta}^2)$$

The equations of motion of the system under an impressed couple (H) are

$$\left.\begin{aligned}
m(a^2+K^2)\ddot{\theta}+K_\theta . \dot{\theta}+\mu\theta+ma . \ddot{y}&=(H)\\
ma . \ddot{\theta}+m\ddot{y}+K_y . \dot{y}+\imath y&=0\\
m\ddot{x}+K_x . \dot{x}+v . x&=0
\end{aligned}\right\} . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

(b) The Forced Vibrations.

As our object is in the first instance to get the forced vibration in θ, the first and second equations in (22) suffice. The third equation shows that the vibration in x is a free vibration which subsides after the initial stage and ultimately disappears. Thus the equations for the determination of θ are identical in form with (2) when $Q_2=0$.

Let $(H)=e^{ipt}$. The vibrations in θ and y are given by q_1 and q_2 respectively in (3) where

$$\left.\begin{aligned}
\omega_1^2&=\frac{\mu}{m(a^2+K^2)} & \lambda_1&=\frac{K_\theta}{m(a^2+K^2)}.\\
\\
\omega_2^2&=\frac{v}{m} & \lambda_2&=\frac{K_y}{m}\\
\\
\delta&=\frac{m^2a^2}{m . m(a^2+K^2)}
\end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

* Rayleigh's Theory of Sound, Vol. 1, p. 205.

Thus when $(H) = \cos pt$ in (22) we have

$$\left.\begin{array}{l} \theta = \dfrac{S_2}{R} \cos (pt - X + \psi_2) \\[2mm] y = \dfrac{ma \cdot p^2}{R} \cos (pt - X) \end{array}\right\} \quad \cdots \cdots \cdots \quad (24)$$

Let $\omega_1 = 2\pi \cdot n_1$ and $\omega_2 = 2\pi \cdot n_2$. Then n_1 is the frequency of the undamped free vibration of the moving system round an unyielding and flexurally rigid axis AB (Figs. 2 and 3), under the control in AB, and n_2 is the frequency of the undamped
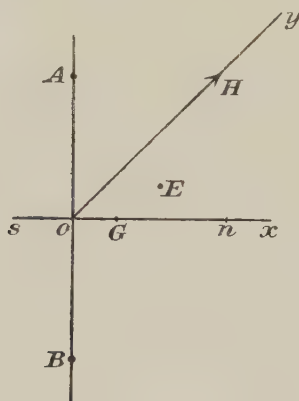
FIG. 2.
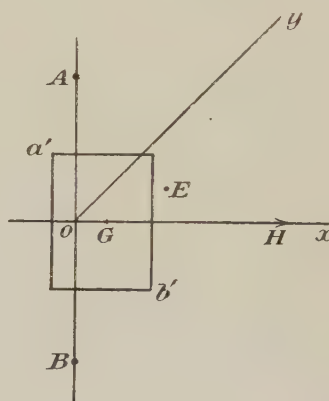
FIG. 3.

free lateral vibration of the wire AB loaded at its middle point O with a mass m equal to the mass of the moving system. The damping factors of these vibrations are $e^{-\frac{1}{2}\lambda_1 t}$ and $e^{-\frac{1}{2}\lambda_2 t}$ respectively.

Let $a = s \cdot k$ where s is a measure of the asymmetry of the moving system. Then

$$\delta = \frac{a^2}{a^2 + K^2} = \frac{s^2}{s^2 + 1} \quad \cdots \cdots \cdots \cdots \quad (25)$$

(c) The Free Vibrations.

The vibration in x is given by the third of equations (22). The vibrations in which θ and y are concerned are determined by the first two of (22) with $(H) = 0$ which lead to the following :—

$$\{(\omega_1^2 - p^2)^2 + \lambda_1^2 p^2\} \{(\omega_2^2 - p^2)^2 + \lambda_2^2 p^2\} - 2\delta p^4$$
$$\{(\omega_1^2 - p^2)(\omega_2^2 - p^2) - \lambda_1 \lambda_2 p^2\} + \delta^2 p^8 = 0 \quad \cdots \quad (26)$$

giving $p/2\pi$ the frequencies of the free vibrations.

(d) Application to the Observations represented in Fig. 1.

The sensitivity at different frequencies from 106 to 126 per second of the gal-

vanometer used by Rosa and Grover is given in Fig. 1. The ordinates of this curve were measured and the results are entered in column (3) of Table I. with the corresponding frequencies in column (1). The curve shows that the amplitude is a minimum at the frequency 114·3 per second. Thus from (16) we have $x_2 = 0$ when $p = 2\pi \times 114 \cdot 3 = \omega_2$. The amplitude is a maximum at the frequencies 110·6 and 120·3

Let $p_1 = 2\pi \times 120 \cdot 3$ and $p_2 = 2\pi \times 110 \cdot 6$. By (16) we get

$$\left(\frac{\omega_1^2}{p_1^2} - 1\right)\left(\frac{\omega_2^2}{p_1^2} - 1\right) = \left(\frac{\omega_1^2}{p_2^2} - 1\right)\left(\frac{\omega_2^2}{p_2^2} - 1\right) = \delta$$

From this and (25) since $\omega_2 = 2\pi \times 114 \cdot 3$ we get ω_1, δ and s.

Thus

$$\left.\begin{aligned} \omega_1 &= 2\pi \times 116 \\ \omega_2 &= 2\pi \times 114 \cdot 3 \\ \delta &= 6 \cdot 82 \times 10^{-3} \\ s &= \frac{1}{12} \text{ very nearly} \end{aligned}\right\} \quad \ldots \ldots \ldots \text{(A)}$$

The ratio of the amplitude at the frequency 120·3 to the amplitude at 110·6 is by Table I. $= \dfrac{46 \cdot 8}{44 \cdot 0} = 1 \cdot 064$. At these resonance frequencies the formula for the

amplitude is (17). Equating the ratio given by (17) to 1·064 and evaluating X_1 and X_2 at these frequencies, we get an equation involving λ_1 and λ_2 which yields

$$\lambda_1 = 4 \cdot 05 \lambda_2$$

This enables us to express the ratio of the amplitudes given by the formula S_2/R in (24) at two selected frequencies in a form involving only one of the λ's—say λ_2. Hence equating the ratio thus expressed to the " observed ratio " derived from Fig. 1 or Table I., we obtain an equation giving λ_2.* Thus, from the amplitudes

at 110·6 and 106, $\lambda_2^2 = 16 \cdot 9$

and at 120·3 and 126, $\lambda_2^2 = 17 \cdot 9$

.·. mean $\lambda_2^2 = 17 \cdot 4$.

Hence we obtain

$$\left.\begin{aligned} \lambda_1 &= 16 \cdot 90 \\ \lambda_2 &= 4 \cdot 17 \end{aligned}\right\} \quad \ldots \ldots \ldots \text{(B)}$$

With the constants given in (A) and (B) the amplitude of the vibration in θ has been calculated from the following relation

$$\text{amplitude} = \text{constant} \times S_2/R,$$

the constant being such a value as to make the calculated amplitude at 120·3 equal to 46·8, the " observed " amplitude. The values thus calculated are given in column (2) of Table I.

* In selecting two frequencies it is advisable to choose a resonating frequency as one. At such a frequency the amplitude is largely determined by the corresponding λ. At frequencies well removed from resonance the amplitude is only slightly affected by the values of the λ's, the term $(x_1 x_2 - \delta)^2$ in (5) being much greater than the others.

TABLE I.

Frequency =n.	Amplitude.			X	ψ_2	$X-\psi_2$
	Calculated.	Observed.	Calc-obs.			
(1)	(2)	(3)	(4)	(5)	(6)	(7)
				،	○	○
106	10·0	9·9	0·1	12·0	2·2	9·8
108	15·2	15·3	−0·1	18·9	2·9	16·0
110	35·9	35·0	0·9	53·1	4·3	48·8
110·6	44·0	44·0	0·0	94·0	5·0	89·0
112	14·2	16·0	−1·8	160·1	8·1	152·0
114	1·6	2·0	−0·4	178·2	47·9	130·3
114·3	—	—	—	178·6	90·0	88·6
116	5·6	5·6	0·0	185·5	168·9	16·6
118	15·5	15·5	0·0	198·2	174·8	23·4
120	44·5	45·6	−1·1	250·6	176·6	74·0
120·3	46·8	46·8	—	267·2	176·8	90·4
122	25·4	24·3	1·1	326·1	177·4	148·7
124	14·0	13·5	0·5	341·6	178·0	163·6
126	9·7	9·8	−0·1	347·1	178·3	168·8

The agreement with the observed values is satisfactory and gives weight to the deductions enumerated below to which we have been led by application of the theory to the observations represented in Fig. 1.

(i)   The frequency of the free undamped lateral vibrations of the suspension AB (Fig. 2) in the direction Oy is 114·3 per second, the wire being loaded with a mass equal to that of the moving system.   The damping factor of this vibration is $e^{-2 \cdot 09t}$, which means that the amplitude diminishes during each second to 13 per cent. of its value at the beginning of the second.

(ii)   The frequency of the free undamped angular vibrations of the system about a flexurally rigid axis coinciding with the axis of suspension and under the control in this axis is 116 per second.   The damping factor of this vibration is $e^{-8 \cdot 45t}$. The meaning of this can best be realised by stating that the amplitude diminishes by 7 per cent. of its value in each period.

(iii)   The centre of mass G of the magnet ns (Fig. 2) is at a distance from the axis of suspension AB equal to 1/12th of the radius of gyration of ns about an axis through G parallel to AB.   In arriving at this deduction it should be remembered that we have assumed that the effective mass of the wire AB is negligible compared with the mass of the magnet.

The frequencies of the free vibrations of the system involving θ and y are obtained by solving* equation (26) in which the constants $\omega_1 \ldots \delta$ have the values given in (A) and (B).   The frequencies obtained are given under (2) in Table II.   In column (3) of the same table will be found the values of a in the damping factor e^{-at} of the corresponding vibration.

TABLE II.

(1) $\omega/2\pi = n$	(2) Free Frequency	(3) a
$n_2 = 114 \cdot 3$	110·6	4·32
$n_1 = 116$	120·3	6·33

* For method solution *see* G. W. Pierce's " Electric Waves and Oscillations," Ch. 8 & 9.

If we put $\lambda_1 = \lambda_2 = 0$ in (26) and solve, we obtain 110·6 and 120·3 for the free frequencies in this case also. This might have been expected because if the dissipative forces be comparatively small the periods of the normal vibrations are the same as if the dissipative forces were absent.

V. Effect of Asymmetry in Moving System on Sensitivity.

If q_1 denotes the co-ordinate of the moving system of a vibration galvanometer whose amplitude is observed, it will be evident from what has been given above that asymmetry introduces a term of the form $\dot{q}_1 \dot{q}$ into the expression for the kinetic energy of the system. Asymmetry may arise from errors of construction, and it is of practical importance to estimate the effect it has on sensitivity. The point has been partially considered in § III. (ii) ; a fuller discussion is given below.

We will take the case where ω_2 and p are kept constant, and resonance is obtained by altering ω_1, this being the case of most frequent occurrence in practice. The conditions for and the amplitude at resonance for a system with two degrees of freedom are given in (10) and (11). But when by a change in the constitution of the system we make $\delta = 0$, then in general the " inertia constants " a_1 and a_2 will also be altered, and (13) is not directly applicable. Let their values become $_0a_1$ and $_0a_2$. Then relation (12) becomes

$$A_0 = \frac{1}{_0a_1 \, \lambda_1 \, p}$$

and instead of (13) we shall have

$$A\delta : A_0 = \frac{_0a_1}{a_1} \left(X_2^2 + \frac{\lambda_2^2}{p^2} \right) \quad : \quad \left(X_2^2 + \frac{\lambda_2^2}{p^2} + \frac{\lambda_2}{\lambda_1}\delta \right). \quad \cdot \quad \cdot \quad \cdot \quad (27)$$

In the case considered in § IV., symmetry can be restored by making $OG = 0$ (Fig. 2), m and k remaining unchanged.

Then

$$\frac{_0a_1}{a_1} = \frac{1}{1 + {}^2} = 1 - \delta$$

which is always less than 1.

In general the effect of asymmetry is to reduce the sensitivity in accordance with the expression given in (27). The following special cases may be noticed :—

(i) λ_2 very small compared with λ_1. Then

$$A\delta : A_0 = {}_0a_1 : a_1.$$

(ii) ω_2 and p are far apart. Then X_2 will be large compared with the other quantities in (27), and again

$$A\delta : A_0 = {}_0a_1 : a_1.$$

In the following table the ratio $A\delta/A_0$ is given for the galvanometer considered in § IV. for various values of δ.

(i) In column (2) when the galvanometer is tuned to an impressed frequency of 110·6.

(ii) In column (3) when it is tuned to an impressed frequency of 120·3.

TABLE III.

	1	2	3
s Approx.	δ	$A\delta/A_0$	
		$p/2\pi = 110 \cdot 6$	$p/2\pi = 120 \cdot 3$
1/32	$0 \cdot 1 \times 10^{-2}$	0·95	0·97
1/25	$0 \cdot 2 \times 10^{-2}$	0·90	0·95
	$0 \cdot 3 \times 10^{-2}$	0·86	0·93
	$0 \cdot 4 \times 10^{-2}$	0·82	0·91
	$0 \cdot 5 \times 10^{-2}$	0·79	0·89
	$0 \cdot 6 \times 10^{-2}$	0·76	0·87
	$0 \cdot 7 \times 10^{-2}$	0·72	0·84
	$0 \cdot 8 \times 10^{-2}$	0·69	0·82
	$0 \cdot 9 \times 10^{-2}$	0·67	0·80
1/10	$1 \cdot 0 \times 10^{-2}$	0·64	0·78
	$2 \cdot 0 \times 10^{-2}$	0·47	0·65
	$3 \cdot 0 \times 10^{-2}$	0·38	0·54
	$4 \cdot 0 \times 10^{-2}$	0·31	0·47
	$5 \cdot 0 \times 10^{-2}$	0·26	0·42
	$6 \cdot 0 \times 10^{-2}$	0·23	0·37
	$7 \cdot 0 \times 10^{-2}$	0·20	0·33
	$8 \cdot 0 \times 10^{-2}$	0·17	0·30
	$9 \cdot 0 \times 10^{-2}$	0·15	0·27
1/3	$10 \cdot 0 \times 10^{-2}$	0·14	0·25

As already stated, tuning is supposed to be effected by altering ω_1 alone. Hence in these cases the loss in sensitivity is not serious, provided *s* does not exceed about 1/32, say. This gives an idea of the accuracy in construction necessary to ensure that no material diminution in sensitivity occurs owing to asymmetry. The authors state they used the galvanometer when it was tuned to about 110 generally. Since δ was found to be $0 \cdot 68 \times 10^{-2}$ it will be seen from the table that the sensitivity was about 72 per cent. of what it would have been had the moving system been mounted symmetrically.

It is therefore of practical importance in the construction and mounting of the moving system in a vibration galvanometer, to make certain that the system is symmetrical in these respects with regard to the axis of rotation. A galvanometer can be tested for asymmetry by observing its sensitivity at different frequencies when an alternating current, preferably of sine wave-form and constant amplitude, is passed through it, and drawing the sensitivity curve as in Fig. 1 from the observations. Asymmetry, if present, will be shown by two peaks in the curve, and a region of very low sensitivity somewhere between the two peaks. Its amount can be determined as in § IV. from the curve, and the loss of sensitivity resulting therefrom can be evaluated by the help of (27). It may be advisable to obtain more than one sensitivity curve for a galvanometer after tuning it to another and a different frequency.

When a constraint is applied so as to keep $q_2 = 0$ at all values of *t* resonance will occur when $\omega_1 = p$ and the amplitude is given by

$$A\delta = \frac{1}{a_1 \lambda_1 p}$$

With $\delta=0$ this would become

$$A_0 = \frac{1}{{}_0 a_1 \lambda_1 \bar{p}}$$

and no constraint would be necessary. Thus $A\delta$ is always less than A_0.

Equations (7) and (8) enable us to determine the force exerted by the constraint, and show that the necessary constraint cannot be obtained by mounting the moving system in a non-uniform or unsymmetrical field on account of the condition $\varphi + \psi_1 = \pi$.

VI. MECHANICAL ANALOGUE OF COUPLED CIRCUITS.

Equations (22) show that in a vibration galvanometer with an asymmetric moving system, such as that considered in §IV., we have a mechanical system capable of motions closely analogous to the electrical vibrations in a pair of coupled circuits. For the mechanical system the equations of motion, when there is no external force impressed on it, may be written

$$\left. \begin{aligned} a \frac{d^2\theta}{dt^2} + \beta \frac{d\theta}{dt} + \gamma\theta + \mu \frac{d^2y}{dt^2} &= 0 \\ \mu \frac{d^2\theta}{dt^2} + a' \frac{d^2y}{dt^2} + \beta' \frac{dy}{dt} + \gamma'y &= 0 \end{aligned} \right\} \quad \cdots \quad (28)$$

Let $R_1 L_1 C_1$ be the resistance, inductance and capacity arranged in series in a circuit in which the current is i_1, and let $R_2 L_2 C_2$ be resistance, &c., in a second circuit in which the current is i_2. Let M be the mutual inductance between the circuits. Then we have for the currents i_1 and i_2 the following equations :—

$$\left. \begin{aligned} L_1 \frac{d^2 i_1}{dt^2} + R_1 \frac{di_1}{dt} + \frac{1}{C_1} i_1 + M \frac{d^2 i_2}{dt^2} &= 0 \\ M \frac{d^2 i_1}{dt^2} + L_2 \frac{d^2 i_2}{dt^2} + R_2 \frac{di_2}{dt} + \frac{1}{C_2} \cdot i_2 &= 0 \end{aligned} \right\} \quad \cdots \quad (29)$$

The correspondence between the quantities involved in (28) and (29) is shown in the following table where the mechanical quantity is placed immediately underneath the corresponding electrical quantity :—

$$L_1 \quad R_1 \quad \frac{1}{C_1} \quad M \quad L_2 \quad R_2 \quad \frac{1}{C_2} \quad \text{electrical.}$$
$$a \quad \beta \quad \gamma \quad \mu \quad a' \quad \beta' \quad \gamma' \quad \text{mechanical.}$$

By attaching the moving system to a bifilar suspension, the tension and the distance apart of whose fibres are separately adjustable, and by making s variable, it would be possible to alter the periods ω_1 and ω_2 and the asymmetry separately, and thus effect changes in the mechanical system corresponding closely to changes in L, C and M in the electrical case.

VII. CONCLUSION—SUMMARY.

After the work detailed above had been completed, my attention was called to a Paper by Kennelly and Taylor entitled : "Some Properties of Vibrating Telephone Diaphragms" (Proc. American Phil. Soc., Vol. 55, pp. 415-46, 1916).

In this Paper the authors describe phenomena observed by them in irregularly-clamped telephone diaphragms exactly analogous to those observed by Rosa and Grover in their vibration galvanometer, represented in Fig. 1. They trace the observed abnormal behaviour of such diaphragms to "the presence of local and loosely-clamped diaphragm boundary areas having a natural frequency independent of the main diaphragm," and draw the inference that "it would only be necessary to attach a local vibrational system . . . to a properly-clamped diaphragm in order to simulate the . . . behaviour of an irregularly-clamped diaphragm." The description of an experimental test which confirms this inference follows. In an appendix to their Paper they give a "provisional theory which was arrived at by searching for a quantitative expression that would satisfy" the observations, and they remark that "it bears a close analogy to the theory of alternating-current coupled circuits."

It will be seen that the authors have anticipated the original purpose of the present Paper. The theory is, however, presented in a form very different from that adopted here. It is formulated with reference to, and its applicatioins directed to the subject of the Paper. It seems, therefore, that the mode of presentation and the application of the theory made in the present Paper, as well as the more searching test of its indications given above, may be of service. The interpretation of, and the derivation of the mechanical characteristics of the system from, the sensitivity curve given in §IV. and the effect of asymmetry on the sensitivity discussed in §V. do not appear to have hitherto received much attention, and may be of value.

IX. *The Relation between Molecular and Crystal Symmetry as shown by X-Ray Crystal Analysis.* *By* G. Shearer, *M.A., B.Sc.*

ABSTRACT.

The methods of X-ray analysis enable the number of molecules associated with the unit cell to be determined. With the help of this information an attempt is made to connect the symmetry properties of the crystal with this number and with the symmetry properties of the molecules from which the crystal is formed.

The symmetry number for each of the 32 crystal classes is given, and is shown to mean the minimum number of asymmetric molecules necessary in the unit cell to satisfy the symmetry conditions. The relative orientations and positions of these molecules in the cell are discussed.

It is suggested that this symmetry number is the actual number of molecules in the cell when the molecule is asymmetric ; further that, if the molecule possesses symmetry, this symmetry appears also in the crystal, and the number of molecules in the unit cell is obtained by dividing the symmetry number of the crystal by the symmetry number of the molecule.

Evidence is produced in support of these hypotheses and examples are given of their application to inorganic and organic crystals.

It has long been known that the various crystals can be divided into 32 classes distinguished from one another by the symmetry which they possess. Much geometrical work has been done on these crystal forms* but, so far, little progress has been made towards a physical interpretation of these geometrical properties. Recent work by the methods of X-ray analysis has thrown much light on the nature of the elementary lattice and the structure of molecules and has provided a basis for an advance in this direction.

The elementary lattice is the smallest which, repeated through space, will build up the crystal. As each lattice is identical with all others, it must show in itself all the symmetry properties of the crystal. These symmetry properties belong to three classes, (1) those associated with planes of symmetry, (2) those associated with axes of symmetry, (3) those associated as an axis of compound symmetry or · a centre of symmetry.

Bravais† in 1850 showed that there are 14 space lattices which in themselves show the full symmetry of the various crystal classes. All of these lattices do not, from an X-ray point of view, appear to be independent. Thus, Bravais differentiates between the simple cube, the centred cube and the face-centred cube. These may all be regarded as one or more interpenetrating simple cubes. The same applies to the centred lattices in the orthorhombic and tetragonal classes. If these are eliminated, there are left nine different lattices, one each associated with the triclinic, hexagonal, tetragonal and cubic classes, two each with the monoclinic, orthorhombic, and trigonal systems. The hexagonal lattice is identical with one of the

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* For a full account of the mathematical theory of the space lattice and its symmetry properties, *see* H. Hilton's " Mathematical Crystallography," Oxford (1903).

† Bravais, Paris Ecole Polytechn. Journal, 19, 1-128 (1850).

two trigonal lattices. There have thus been eliminated from consideration all lattices which show more than one molecule associated with the unit cell. The view here adopted of an elementary lattice is that of a parallelopiped which, with perfectly symmetrical molecules placed at its vertices, shows the full symmetry of the class of crystal with which it is associated. It will be shown that by a combination of a definite number of similar interpenetrating lattices the symmetry of any class can be obtained even if the molecule possesses no symmetry of its own.

First of all, it is necessary to define what is meant by a plane or axis of symmetry in the elementary lattice. Take the parallelopiped which is the elementary lattice of the crystal class (Fig. 1) and consider molecules placed at its vertices. As the lattice has to repeat through space all these molecules must be orientated in exactly the same way relative to the lattice. Consider, for example, the vertex A of the lattice. At A eight lattices meet and the molecule at A belongs equally to all of them. The molecule* A of this cell corresponds to the molecules $B, C, D, E,$
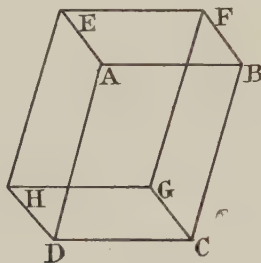
Fig. 1.

F, G, H of the other cells which meet at A. It follows that all the molecules at the vertices must be orientated relative to the lattice in exactly the same way.

Suppose the crystal has a plane of symmetry parallel to the face A, B, C, D. There must exist a set of molecules—" B " molecules—which are the mirror images of the molecules at the vertices—the " A " molecules. As these cannot coincide with the " A " molecules unless the molecule mirror images into itself there must be a separate lattice precisely similar in shape to the first but displaced in some way relative to the first lattice. It is clear that, to pass from an " A " molecule to a " B " molecule, we must take the mirror image of the "A " molecule in some plane parallel to the plane of symmetry and then give it a translation to some point in the cell. A reflection in the same plane and an equal translation applied to any other " A " molecule must give rise to another " B " molecule. Fig. 2 shows these two interpenetrating lattices, the arrow-heads denoting the orientation of the molecules.

Exactly the same process applied to the " B " molecules must give rise to " A " molecules. This reciprocal relation puts certain restrictions on the relative positions of the " A " and " B " molecules in the cell. These will be considered later.

* The term "Molecule" is used in this Paper to denote "Crystal Molecule." It does not follow that the arrangement of atoms in it is identical with the arrangement in the molecule in solution as dealt with in chemistry. There appears, however, to be a strong resemblance between the two in many cases. It has, of course, the same component atoms and molecular weight.

In the same way, the existence of a digonal axis gives rise to a second set of molecules whose orientations relative to the " *A* " molecules are obtained by rotating these about the axis through 180°. Once more, to pass from the one set to the other, in addition to the rotation there must be a translation and this translation must be such that the operation which, applied to the first set, gives the second, will, applied to the second set, give the first. Similar relations hold for axes other than digonal.

Having now defined what we mean by the elementary cell and planes and axes of symmetry, we may now proceed to determine the number of interpenetrating simple lattices, or, in other words, the number of molecules in the elementary cell which are necessary to satisfy the symmetry properties of the 32 crystal classes. First of all, let it be assumed that the molecule from which the crystal is constructed possesses no symmetry of its own. It is clear that the existence of a digonal axis must give rise to a second molecule in the cell whose orientation
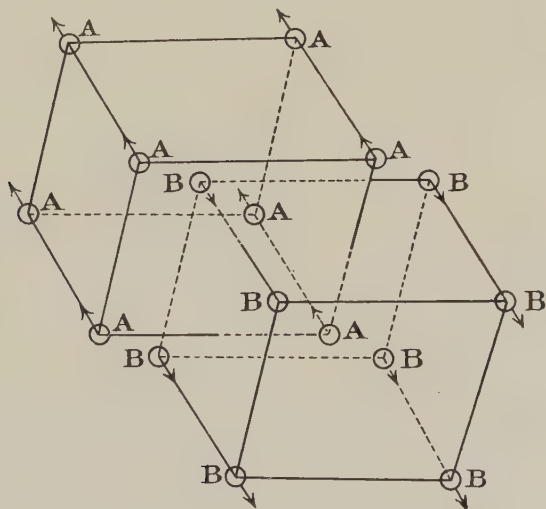
Fig. 2.

relative to the first is obtained by a rotation of 180° about the axis. A trigonal axis will give rise to two other molecules in the cell whose orientations are obtained by rotations of the first through angles of 120° and 240° about the axis. In the same way a plane of symmetry will give rise to a second which is the mirror image of the first in the plane of symmetry. A combination of a plane of symmetry and a digonal axis at right angles to it will give an elementary cell with four differently orientated molecules—" *A*," " *B*," " *C* " and " *D* " molecules. The orientation of the " *B* " molecules is obtained by reflection in the plane of symmetry, that of the " *C* " molecules by rotation of the " *A*'s " about the axis, while that of the " *D* " molecules may be obtained either by a rotation of the " *B* " molecules or by a reflection of the " *C* " molecules.

In this way it is possible to construct a table of the 32 classes and tabulate the minimum number of asymmetric molecules which must necessarily appear in the elementary cell in order to satisfy the symmetry properties of the crystal. This

number is most readily obtained by taking the stereographic projection of the crystal and finding the number of points on the diagram corresponding to the general face. This number may conveniently be called the " Symmetry Number " of the crystal class.

TABLE.

Crystal class.		System.	Class.	Symmetry number or number of asymmetric mole- cules per cell.
No.	Schönfliess symbol.			
1	C_1	Triclinic ...	Triclinic Asymmetric	1
2	S_2		,, Pinakoidal	2
3	S	Monoclinic	Monoclinic Domal	2
4	C_2		,, Sphenoidal	2
5	C_2^h		,, Prismatic	4
6	V	Orthorhombic	Rhombic Bisphenoidal	4
7	C_2		,, Pyramidal	4
8	V^h		,, Bipyramidal	8
9	C_4	Tetragonal ...	Tetragonal Pyramidal	4
10	S_4		,, Bisphenoidal	4
11	D_4		,, Trapezohedral	8
12	C_4^h		,, Bipyramidal	8
13	C_4^v		Ditetragonal Pyramidal	8
14	S_4^u		Tetragonal Scalenohedral	8
15	D_4^h		Ditetragonal Bipyramidal	16
16	C_3	Trigonal ...	Trigonal Pyramidal	3
17	C_3^i		,, Rhombohedral	6
18	D_3		,, Trapezohedral	6
19	C_3^h		,, Bipyramidal	6
20	C_3^v		Ditrigonal Pyramidal	6
21	D_3^d		,, Scalenohedral	12
22	D_3^h		,, Bipyramidal	12
23	C_6	Hexagonal ...	Hexagonal Pyramidal	6
24	D_6		,, Trapezohedral	12
25	C_6^h		,, Bipyramidal	12
26	C_6^v		Dihexagonal Pyramidal	12
27	D_6^h		,, Bipyramidal ...	24
28	T	Cubic ...	Tetragonal Pentagonal Dodecahedral	12
29	O		Pentagonal Icositetrahedral ...	24
30	T^h		Dyakis Dodecahedral	24
31	T^d		Hexakis Tetrahedral	24
32	O^h		Hexakis Octahedral	48

It has already been mentioned that it is possible to lay down certain restrictions as to the relative positions of the variously orientated molecules in the lattice. Take, for example, a crystal with a plane of symmetry perpendicular to the " b " axis of the crystal. Place " A " type molecules of the verticles of a parallelopipedon of sides a, b, c, with angles 90°, β, 90°—one of the monoclinic lattices. Suppose that the molecule at the origin may be represented by the set of points

$$x_1, \ y_1, \ z_1$$

$$z_2, \ y_2, \ z_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$z_n, \ y_n, \ z_n$$

All the molecules of type " A " in the crystal may then be represented by the set

$$x_1 + la, \; y_1 + mb, \; z_1 + nc$$
$$x_2 + la, \; y_2 + mb, \; z_2 + nc$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$x_n + la, \; y_n + mb, \; z_n + nc$$

where l, m, n are any positive or negative integers.

A reflection of these molecules in the plane $y = 0$ combined with a translation aa, βb, γc parallel to the three axes gives the set of " B " molecules,

$$x_1 + la + aa, \; -y_1 - mb + \beta b, \; z_1 + nc + \gamma c$$
$$x_2 + la + aa, \; -y_2 - mb + \beta b, \; z_2 + nc + \gamma c$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$x_n + la + aa, \; -y_n - mb + \beta b, \; z_n + nc + \gamma c$$

The same operation applied to this set must bring it into coincidence with the " A " set. If we carry out this operation we obtain the set,

$$x_1 + la + 2aa, \; y_1 + mb, \; z_1 + nc + 2\gamma c$$
$$x_2 + la + 2aa, \; y_2 + mb, \; z_2 + nc + 2\gamma c$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$x_n + la + 2aa, \; y_n + mb, \; z_n + nc + 2\gamma c$$

For these to represent the " A " set, we must have

$$2aa = O \text{ or } a \text{ or } 2a, \text{ &c., and } 2\gamma c = O \text{ or } c \text{ or } 2c, \text{ &c.}$$

Hence $\qquad a = O \text{ or } \tfrac{1}{2}, \text{ and } \gamma = O \text{ or } \tfrac{1}{2}$

Hence the " B " molecules must lie on lines parallel to the " b " axis either coinciding with the edges of the lattice or passing through the middle points of the sides or faces.

Consider now a digonal axis of symmetry parallel to the " b " axis. A rotation of the " A " molecules through 180° combined with a translation $a'a$, $\beta'b$, $\gamma'c$ parallel to the axes gives the set of " C " molecules

$$-x_1 - la + a'a, \; y_1 + mb + \beta'b, \; -z_1 - nc + \gamma'c$$
$$-x_2 - la + a'a, \; y_2 + mb + \beta'b, \; -z_2 - nc + \gamma'c$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$-x_n - la + a'a, \; y_n + mb + \beta'b, \; -z_n - nc + \gamma'c$$

The same operation applied to this set must bring it into coincidence with the " A " set. This gives the set

$$x_1 + la, \; y_1 + mb + 2\beta'b, \; z_1 + nc$$
$$x_2 + la, \; y_2 + mb + 2\beta'b, \; z_2 + nc$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$x_n + la, \; y_n + mb + 2\beta'b, \; z_n + nc$$

Hence, we must have $\qquad \beta' = O \text{ or } \tfrac{1}{2}$

Hence, the "*C*" molecules must lie either in the *a—c* planes or in planes half-way between successive *a—c* planes.

Similar relations may be worked out for the other symmetries which the crystal may possess. For example, the existence of a trigonal, tetragonal or hexagonal axis involves the existence of three, four or six molecules, derived from each other by successive rotations of 120°, 90°, or 60° about the axis. The projections of corresponding points of these on a plane perpendicular to the axis form an equilateral triangle, square or regular hexagon according to the nature of the axis. Their distribution in a direction parallel to the axis is such that either they all lie in the basal planes or are equally distributed between successive basal planes. In the latter case the successive molecules form a regular spiral which may be right or left-handed. The simultaneous existence of a plane of symmetry passing through the axis or perpendicular to it gives rise to a second spiral which is in the opposite sense to the first. The position of this spiral relative to the first one obeys the conditions already worked out as we may consider the whole of the first spiral as a more complicated molecule.

In this connection it is interesting to note that the rotation of the plane of polarisation of light by a crystal is intimately associated with the existence of this spiral arrangement and the dextro or lævo nature of the rotation depends on the type of spiral. If any spiral has this property it is clear that the existence of a plane of symmetry introduces a spiral which will give an equal and opposite rotation and the crystal will appear inactive. This is, of course, shown by the fact that the only crystals which show this rotation are those whose only symmetry is associated with axes of symmetry. A digonal axis is, of course, inactive as it is impossible to distinguish between a rotation of 180° in a right or left-handed direction. Any biaxial crystal which rotates the plane of polarisation must do so owing to the existence of some spiral arrangement within the molecule itself.

In this way we can go a very considerable distance towards the solution of the crystal structure of any crystal using only the ordinary crystallographic data and, in the case of a symmetrical molecule, the number of molecules in the unit cell. Certain quantities remain to be determined. These are such quantities as β, a', γ' in the above discussion and the sides of the equilateral triangle, square or hexagon in the case of uniaxal crystals.

So far it has been assumed that no symmetry exists in the molecule. As long as an asymmetric molecule is in question the number of molecules in the unit cell must be that given in the table or an integral multiple of it. If it were less, the crystal could not possibly have its full symmetry.

If, however, the molecule possesses a certain amount of symmetry of its own, a number less than that given in the table may suffice. Thus a molecule with a plane of symmetry may form a crystal of, say, class 5 with only two molecules per cell, the second being derived from the first by a rotation about the digonal axis. This can only happen, however, if the plane of symmetry of the molecule is set in the crystal parallel to the plane of symmetry of the crystal, in this case a plane perpendicular to the "*b*" axis.

The converse of this propostion is of importance. If it is found that the number of molecules in the unit cell is less than the symmetry number for the crystal, it follows that the molecule must possess some symmetry and that this symmetry is reproduced in the crystal. As the number of molecules in the elementary cell

must be an integer, it follows that the symmetry which the molecule shares with the crystal must be represented by an integer which is obtained by dividing the number given in the table corresponding to this class of crystal by the number of molecules in the unit cell. Thus, if n is the number characteristic of the class and m the number per unit cell, then n/m is an integer and the crystal shares n/m fold symmetry with the molecule. This number may be called the Symmetry Number of the molecule.

So far it has not been claimed that the number of molecules given in the above table is the actual number present when the molecule is asymmetric, but only that the number in the cell must be either this number or an integral multiple of it. It naturally suggests itself that this number is the actual number. This means that the number of molecules in the elementary cell is the minimum necessary to satisfy the symmetry conditions. It seems natural to suppose that a crystal will be constructed with the minimum material. If this is true, it implies that each unit cell contains one and one only of each differently orientated molecule.

A very considerable amount of experimental evidence exists in support of this hypothesis. In testing it, attention must be confined to asymmetric molecules. W. H. Bragg* has recently investigated a number of aromatic organic compounds, and many of these have molecules which can possess little or no symmetry. A large percentage of these form monoclinic prismatic crystals and show four molecules in the elementary cell. This is what is to be expected if the assumption made above is true. So far no example has been found where the number is greater than the symmetry number of the crystal. Certain cases have been found where the number is less than the symmetry number. This must mean that the molecules themselves possess some symmetry which they share with the crystal. In all the cases where this occurs there is good reason to suppose that the molecule does possess symmetry. For example, both naphthalene and anthracene form monoclinic crystals with two instead of four molecules in the unit cell. This shows that these molecules both possess a twofold symmetry. The use of such information will be discussed later. Both α- and β-naphthol belong to the same class, but have four molecules in the elementary cell. It is only to be expected that the addition of an OH group to one side of the naphthalene double ring will destroy any symmetry which the naphthalene ring possessed. In the same way the addition of the COOH, &c., groups in such substances as benzoic, salicylic and phthalic acids apparently destroys the symmetry of the benzene molecule, as these compounds all form monoclinic prismatic crystals with four molecules per cell. Tartaric acid is a monoclinic sphenoidal crystal and, as is to be expected, has two molecules in the unit cell.

As no case has occurred which contradicts the hypothesis, and as there is a considerable mass of evidence in its favour, it seems reasonable to conclude that it is generally true that the number of molecules in the elementary cell is always the minimum number necessary to impart to the crystal the necessary symmetry properties.

The question of molecules which do possess symmetry must now be considered. Here again a natural hypothesis suggests itself, viz., that all the symmetry of the molecule is reproduced in the crystal. It seems reasonable to assume that nature in forming a crystal will make use of all the symmetry already existing in the molecule

* Bragg, Proc. Phys. Soc., Vol. 34, p. 33 (1921).

Indeed, this is really the same assumption as has already been made in the discussion of asymmetric molecules. It simply amounts to the statement that the minimum amount of material is used in the building up of a crystal. If this hypothesis is true it follows that the crystal must always show at least as much symmetry as the molecule from which it is formed. Indeed, there is evidence in favour of the fact that in general the symmetry of the crystal is higher than that of the molecule.

In the present state of our knowledge of molecular structure it is more difficult to test this assumption. There is no doubt that much of the symmetry is certainly made use of, but it is impossible to say at present exactly what symmetry any particular molecule possesses. There seems, however, to be considerable evidence that all the symmetry of the molecule is used in the formation of the crystal. The fact that most of the elements form crystals of a high order of symmetry with relatively few molecules in the unit cell shows that a considerable amount, if not all, of the symmetry of the atoms is reproduced in the crystal. The same applies to many simple inorganic salts. For example, $CaCO_3$ and Al_2O_3 are both likely to possess a trigonal axis, the oxygen atoms being arranged trigonally round the rest of the molecule. A calculation of the symmetry of the molecule from the number of molecules in the unit cell and the nature of the crystal shows that the molecules must possess a trigonal axis. Quartz forms a crystal of class 18 with three molecules in the elementary cell. This implies that the SiO_2 molecule itself possesses a digonal axis and that the molecules are so arranged in the crystal that this digonal axis of the crystal coincides with one of the digonal axes of this type of crystal. So far no case has been investigated which does not seem to agree with the assumption that all the symmetry of the molecule is reproduced in the crystal.

If this hypothesis is true, it should be of great service in determining the structure of molecules. If, in addition to being able to say that a molecule possesses such and such symmetry we can also say that it possesses no other symmetry, we are in a strong position to decide on the nature of the molecule and also on the question of what symmetry in a molecule really means. In dealing with an atom it is clear that the outer electrons must be the important factors in deciding the crystal structure and the information to be derived from such arguments as those given above should throw considerable light on the distribution of these outer electrons. It is immaterial whether the electrons are considered as stationary or as moving in orbits. In the latter case the distribution of the orbits or of the normals to the orbits must show the symmetry properties.

Our knowledge of the distribution of the outer electrons is still too vague to permit of a complete interpretation of the results which the information derived from crystallographic measurements gives, but certain steps may be made in this direction. For example, many of the elements crystallise in the highest form of cubic symmetry and possess two or four atoms in the unit cell. This means that the atoms in question possess either a 24-fold or a 12-fold symmetry. An examination of the table suggests that the symmetry of the atoms is that of one of the lower forms of cubic symmetry. This leads to the conclusion that there is some form of cubic arrangement of the electrons or of the electron orbits. Similar conclusions as to the existence of some such cubic arrangement have been reached as a result of the consideration of the chemical properties of various atoms. Carbon in the form of the diamond is a crystal of class 31 with eight atoms in the unit cell. This fact ascribes to the carbon atom a symmetry number 3—that is to say, a trigonal axis.

Bohr has arrived at the conclusion that the four outer electrons in the carbon atom move in four 2_1 orbits whose normals are arranged tetrahedrally. Such an arrangement provides a ready explanation of the existence of a trigonal axis in the atom, but would also suggest the existence of other trigonal axes of symmetry. It may, of course, be necessary also to consider the two inner electrons, or the arrangement may not be perfectly tetrahedral. The structure of graphite is still somewhat uncertain. Debye and Scherrer* found that their observations agreed with the assumption of a rhombohedral lattice with eight atoms in the unit cell. On the other hand, Hull† suggests four interpenetrating hexagonal lattices. A consideration of the numbers given in the table for the various classes of trigonal crystals shows that if the above assumptions are true it is impossible to have eight atoms in the unit cell. The same objection does not apply to Hull's suggested structure. Hull's structure supports the idea that the carbon atom in graphite possesses a trigonal axis just as it does in the case of diamond.

A few typical examples of the application of these principles may now be con-
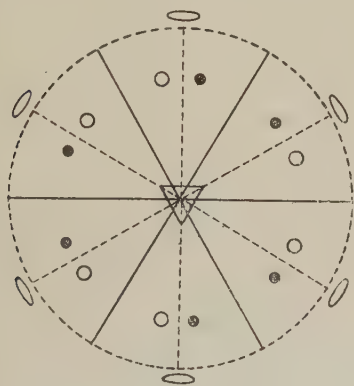
FIG. 3.
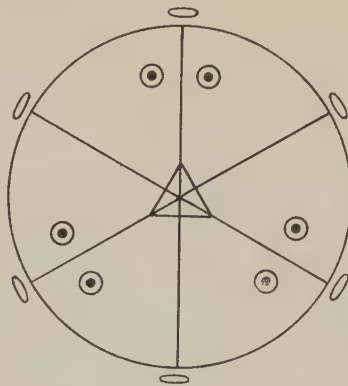
FIG. 4.

sidered in more detail. Take first the case of ruby—Al_2O_3. This forms a crystal of class 21, possessing a trigonal axis, three digonal axes inclined at angles of 120 deg. to each other and perpendicular to the trigonal axis, and three planes of symmetry passing through the trigonal axis and bisecting the angles between successive digonal axes. The stereographic projection of the crystal class is shown in Fig. 3.

How much of this symmetry can the Al_2O_3 molecule possess? If the two aluminium atoms are placed along the trigonal axis and the oxygens are placed at the corners of an equilateral triangle in the equatorial plane of the two aluminiums, a highly symmetrical molecule is formed. If the component atoms are considered as spheres, the symmetry of the molecule is that shown in Fig. 4. It has a trigonal axis, three digonal axes, three planes of symmetry each passing through the trigonal axis and one of the digonal axes and an equatorial plane of symmetry. Although this molecule possesses, like the crystal, both planes and axes of symmetry, the distribution of these is not that of the crystal. In fact, Fig. 4 corresponds

* Debye and Sherrer, Phys. Zeit., 13, 297 (1917).

† Hull, Phys. Rev., 10, 692 (1917).

to a crystal of class 22. The ruby crystal has planes of symmetry bisecting the angles between the digonal axes. In order to form such a crystal from the above molecule it would be necessary to introduce a second molecule into the lattice, whose orientation is such that it is the mirror image of the first in a plane of symmetry bisecting the angle between two of the digonal axes of the molecule. If this were done the symmetry of the resulting molecule would be that shown in Fig. 5. The
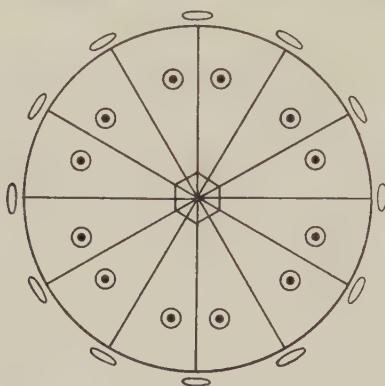
FIG. 5.

crystal would then belong to the dihexagonal bipyramidal class. As observations show that the crystal is ditrigonal scalenohedral and not dihexagonal bipyramidal, it follows that the molecule cannot have all the symmetry attributed to it above. It is clear that the maximum symmetry which the molecule can possess is (1) a trigonal axis, (2) either three digonal axes or three planes of symmetry. It cannos possess both the planes and the axes. It is difficult to decide whether it is the planet
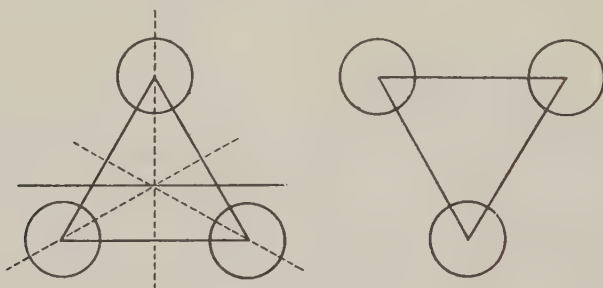
FIG. 6.

or the axes which actually exist in the molecule. It is obvious, however, that it is not possible to consider the various atoms as spheres. The deciding factor in the determination of the symmetry must be the distribution of the outer electrons in the component atoms.

As the moleule cannot possess the full symmetry of the crystal there must be at least two molecules in the unit cell. The second of these is derived from the first

by a reflection in a plane of symmetry or a rotation about a digonal axis according as the molecule possesses in itself the axes or the planes of symmetry. The result will be two sets of Al_2O_3 molecules whose oxygen atoms are arranged as shown in Fig. 6. This is in entire agreement with the results obtained by Bragg* from measurements by X-ray methods.

Quartz (SiO_2) provides another interesting example. It forms a crystal of the trigonal trapezohedral class and has three molecules in the unit cell. If the molecule were asymmetric six molecules would be necessary. Hence the crystal must share with the molecule a two-fold symmetry. Since the crystal possesses no planes of symmetry this common element of symmetry must be a digonal axis. Hence the SiO_2 molecule must be capable of reproducing itself when rotated through 180 deg. about some axis. It must therefore be of some such form as is shown in Fig. 7. Here again, if the atoms are regarded as spheres, the molecule would possess in addition two planes of symmetry. As the crystal does not show these planes it is clear that the outer electrons in the oxygen atoms are so distributed that the molecule will not reflect into itself across a plane bisecting the angle between the two Si—O directions or across a plane passing through the centres of the three atoms. This fact throws further light on the symmetry of the ruby molecule. In that
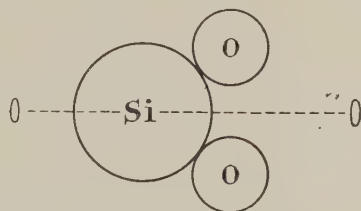
FIG. 7.

case it was not found possible to decide whether the molecule possesses the planes or the digonal axes of symmetry. The fact that the oxygens in the quartz molecule can rotate into each other but not reflect into each other makes it extremely probable that the same is true in the case of ruby, and that therefore the Al_2O_3 molecule possesses three axes but no planes of symmetry.

By placing three SiO_2 molecules in the unit cell, obtained from each other by successive rotations of 120 deg. about the trigonal axis and equally spaced along this axis, and such that the projections of corresponding points of these three molecules on a plane perpendicular to the trigonal axis form an equilateral triangle, all the necessary symmetry is given to the crystal. It is, of course, obvious that the trigonal axis of the crystal must be perpendicular to the digonal axis of the molecule. W. H. Bragg's measurements by X-ray methods result in a model which supports these arguments.

An examination of the crystal models of many simple inorganic compounds fails to reveal the presence of a definite molecule in the crystal. Thus in NaCl each sodium atom is surrounded by six chlorine atoms, and each chlorine atom by six equidistant sodium atoms. The same applies to KCl. In Ice each hydrogen is surrounded by two oxygens, and each oxygen by four hydrogens giving the necessary

* See W. H. and W. L. Bragg, X-Rays and Crystal Structure, 2nd edition.

proportion of hydrogen to oxygen. In fact Bragg* has used this fact to construct a crystal model of ice, and has shown that the result agrees with the X-ray measurements made by Dennison. In more complex compounds the molecule certainly appears in the crystal, and it seems reasonable to assume that even in the above cases the molecule really exists and that, in the case of NaCl for example, each chlorine atom has its own particular sodium. It will be shown that if the molecule does actually exist, the sodium to which any particular chlorine belongs is not one of the six surrounding it at equal distances, but one of the sodiums arranged diagonally relative to the chlorine.

If it is assumed that the molecule does actually exist, even in polar compounds, some interesting results as to the symmetry of various molecules follow.

NaCl and KCl form cubic crystals of classes 32 and 29 respectively. Both have four molecules in the unit cube. The symmetries which the molecules share with the crystals are therefore twelve-fold and six-fold respectively.

Consider the case of KCl. This six-fold symmetry must consist of (1) a trigonal

$\bigcirc$ 2_1 or 3_1
● 2_2 or 3_2

FIG. 8.

$\bigcirc$ 2_1 *Na Atom* $\square$ 3_1 *Cl Atom*
● 2_2 $\square$ 3_2
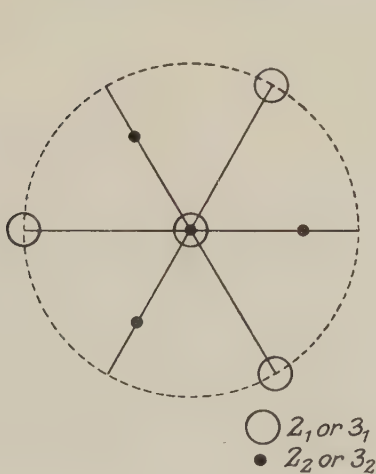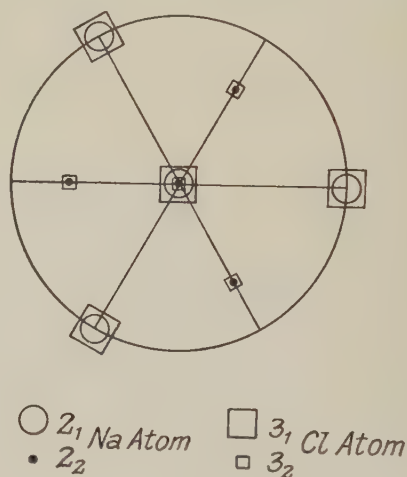
FIG. 9.

axis, and (2) a digonal axis perpendicular to the trigonal axis. Such a symmetry involves the presence of two other digonal axes perpendicular to the trigonal axis, and inclined to each other and to the first digonal axis at angles of 120 deg.

The question arises—how can the KCl molecule possess this symmetry? If the K and Cl atoms are placed on the trigonal axis then there must be three digonal axes perpendicular to the K—Cl line. A rotation of the molecule through 180 deg. about one of these axes brings the K atom to the position of the Cl atom. It follows that the K and Cl atoms must be interchangeable as far as symmetry considerations go. This is not so unreasonable as it appears at first sight when it is remembered that what we have to deal with is a combination of a positively charged K atom with a negatively charged Cl atom. The outermost electron of K atom has passed over into and been absorbed by the Cl atom. Each atom has then the electron

* W. H. Bragg, Proc. Phys. Soc., Vol. 34, p. 98 (1922).

arrangement of the inert gas Argon. They differ only in the existence of positive and negative charges on the two atoms. It is quite conceivable that this difference of charges has no bearing on the question of crystal symmetry, and we may therefore regard the K atom as being able to rotate into the Cl atom without making any difference in the molecule from a crystal point of view.

Similar arguments apply to the case of NaCl. The Na atom becomes a positively charged Neon atom, while the Cl atom becomes a negatively charged Argon atom. It appears that for crystallographic purposes a positively charged Na atom is identical with a negatively charged Cl atom. Such arguments point to the fact that in crystallographic questions the governing factor is not the nature of the nucleus, but the electron arrangement around the nucleus and more especially the arrangement of the outer electrons.

Consider the electron arrangement in these atoms. The following table is that given by Bohr[*] for the numbers of electrons in the various types of orbit :—

TABLE.

Element.	Number of electron orbits of various types.							
	1_1	2_1	2_2	3_1	3_2	3_3	4_1	4_2
Ne ...	2	4	4	—	—	—	—	—
Na ...	2	4	4	1	—	—	—	—
Cl ...	2	4	4	4	3	—	—	—
A ...	2	4	4	4	4	—	—	—
K ...	2	4	4	4	4	—	1	—

Both Neon and Argon possess two sets of outer electrons, four in each set. In Neon, the four 2_1 orbits are such that the normals to their planes are arranged tetrahedrally in space. The four 2_2 orbits are circular. Bohr does not state how these are arranged. The most symmetrical arrangement appears to be obtained by arranging three of the normals in a plane normal to one of 2_1 normals at angles of 120 deg. to each other, and the fourth normal in a direction perpendicular to this plane. The direction cosines of these planes may be represented by

$$\left.\begin{array}{c}111\\1\bar{1}1\\\bar{1}1\bar{1}\\\bar{1}\bar{1}1\end{array}\right\} \text{ for the } 2_1 \text{ orbits and } \left.\begin{array}{c}\bar{1}\bar{1}1\\01\bar{1}\\1\bar{1}0\\\bar{1}01\end{array}\right\} \text{ for the } 2_2 \text{ orbits.}$$

The stereographic projection of these orbits is shown in Fig. 8.

The atom possesses one trigonal axis and three planes of symmetry, but no digonal axes of symmetry. If we consider two of these atoms placed end to end in opposite directions we obtain a molecule which possesses one trigonal axis, three digonal axes and four planes of symmetry as shown in the stereographic projection, Fig. 9.

This is the symmetry of the NaCl molecule. Fig. 10 shows the suggested distribution of electron orbit normals. If we rotate the lower atom of the molecule through some angle about the trigonal axis so as to destroy the planes of symmetry,

[*] Bohr, Zeitschr. fur Physik, 9, 1 (1922).

we obtain a molecule which possesses a trigonal axis and three digonal axes—the necessary symmetry of the KCl molecule. The stereographic projection of such a molecule is shown in the figure given below.

We thus see that by considering the distribution of the electrons round the nuclei we can construct molecular models which have the desired symmetry. It is, of course, not claimed that these figures represent the actual arrangement of the electrons ; many other arrangements could no doubt be constructed. The important thing is that it can be demonstrated that it is the electron distribution around the nuclei which matters and not the nuclei themselves.

Ice forms another interesting example. X-ray data indicate that the crystal is hexagonal with four molecules in the unit cell. An examination of Table I. shows

$\longrightarrow$ *2_1 or 3_1 Normal*
$\multimap$ *2_2 or 3_2* *"*
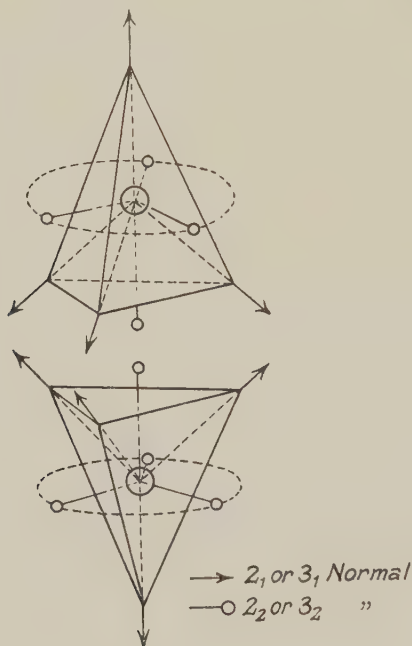
Fig. 10.

that the molecule must possess at least a trigonal axis. This the molecule cannot do unless the three atoms of the molecule are arranged along the trigonal axis. X-ray data show that such is not the case. The hydrogen atom consists of a nucleus with one outer electron ; the oxygen atom has six outer electrons. If the two electrons from the hydrogens join with the six oxygen electrons to form the electron arrangement of neon, what has to be considered is a doubly charged neon atom surrounded by two hydrogen nuclei. It is possible that these nuclei play no part in the determination of the symmetry of the molecule from a crystallographic point of view, and that all that has to be considered is a doubly charged neon atom. In the discussion of the NaCl crystal it has been shown that this atom has probably a six-fold symmetry. As there are four ice molecules in the unit cell it follows that

the crystal should possess 24-fold symmetry. This is in agreement with the assumption of a hexagonal form for the crystal.

It is also of interest that in the case of ice the distance between the centres of the oxygen and hydrogen atoms is 2·76 A.U., a distance considerably greater than the commonly accepted value. This fact may be explained if it is assumed that the hydrogens are playing no part except in determining the residual electric field round the molecule and that what has been measured is the distance between the centres of two negatively charged neon atoms.

In the same way the symmetry of ZnS should be similar to that of KCl. The crystal is cubic (hexakis tetrahedral) and possesses four molecules in the unit cell. The symmetry of the molecule must therefore be six-fold, that is to say, the same as that of the KCl molecule.

There are, of course, many inorganic polar compounds which cannot be considered as an aggregate of inert gas atoms. Such molecules as $CaCO_3$, $NaNO_3$,

$$\bigcirc \, 3_1 \quad \square \, 3_1$$
K Atom   Cl Atom
$$\bullet \, 3_2 \quad \square \, 3_2$$
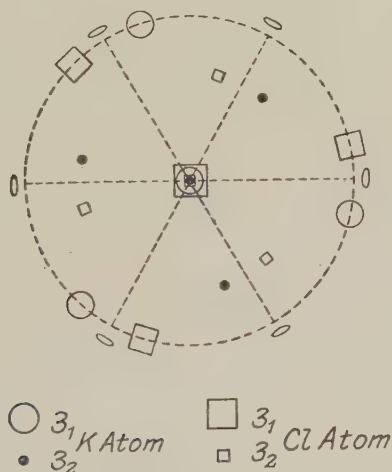
FIG. 11.

$NaClO_3$, &c., cannot be so regarded. The ClO_3 ion has 18 outer electrons which is the number possessed by Argon. The distribution of these electrons is, however, very different from that of the Argon atom, and we cannot consider the two as equivalent in any respect. As is to be expected, crystallographic measurements show that the molecule possesses a trigonal axis but no planes or digonal axes. In general, where two or more different atoms occur in one of the ions, this ion will not become identical in electron arrangement, with an inert gas atom. Hydrogen is anomalous in so far as it possesses only one electron, and when it has lost this electron only the positively charged nucleus remains, and it may be permissible in some cases to ignore the effect of this nucleus in dealing with the symmetry of the molecule from a crystallographic point of view. Ice appears to be an example of this.

The application of these principles to organic crystals is somewhat more limited as the molecules of such compounds themselves possess little or no symmetry. A very large number of the crystals so far investigated are monoclinic prismatic and

possess four molecules in the elementary cell. This would indicate a complete absence of symmetry in the molecules. There are, however, some notable exceptions. For example, both naphthalene and anthracene belong to this class, but have only two molecules in the unit cell. This must mean that these molecules share a two-fold symmetry with the crystal. Benzene forms a rhombic bipyramidal crystal and has two molecules per cell. This gives to the benzene molecule a four-fold symmetry, possibly a plane of symmetry and an axis of symmetry perpendicular to the plane. In view of the fact that the structure of the benzene ring has been the subject of much debate, this new information may be of considerable importance. There have been many suggestions as to the arrangements of the fourth valency bonds, but most of these would appear to give to the molecule a trigonal axis. This is not supported by crystallographic work. Whatever the arrangement of the bonds, the molecule, in its crystalline state at least, must show a plane of symmetry with a digonal axis perpendicular to it or some equivalent symmetry.

Neither the Kekulé nor the Claus formula conforms with this condition. On the other hand, the Ladenburg formula does, and so also does the formula suggested by Dewar. Ingold* has recently shown that this Dewar formula has much chemical evidence to support it. The arrangement of bonds is that shown in Fig. 12. If
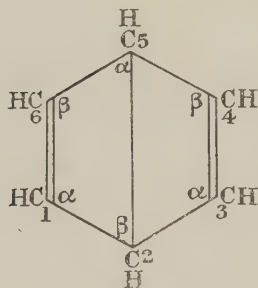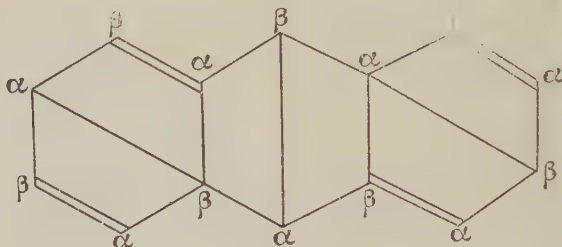
FIG. 12.   FIG. 13.

the puckered hexagon suggested by Bragg† is assumed and the bonds are arranged in this way, it is found that the molecule has the required symmetry. Fig. 12 shows the symmetry of the molecule. The points marked α are supposed to lie above the plane mean plane of the ring and those marked β below the mean plane.

If this is extended to the case of anthracene the molecule has a centre of symmetry, and this is represented in Fig. 13. Naphthalene presents greater difficulties. The X-ray measurements show that the molecule has a twofold symmetry—probably a centre of symmetry. Such formulæ as those of Fig. 14 satisfy this condition. All the formulæ usually given for naphthalene appear to possess too great symmetry. Bragg's measurements of the naphthalene crystal show that there are two molecules in the unit cell and indicate further that the second is derived from the first by a rotation of 180° about the " b " axis. The presence of these two molecules, together with the existence of a centre of symmetry in the molecule, gives to the crystal the full symmetry of the monoclinic prismatic class.

* Ingold, Journal Chem. Soc., June (1922).
† Bragg, Proc. Phys. Soc., Vols. 33, 34 (1921).

An examination of the measurements made by crystallographers on benzene substitution products would appear to indicate that when two of the hydrogens are replaced by the same groups the 1-3 product is a crystal of higher symmetry than the 1-2 or the 1-4 product. This probably implies that the 1-3 compound possesses in itself some symmetry, while the 1-2 and 1-4 compounds are in general asymmetric. If the 1, 2, 3, 4 positions are as shown in Fig. 12, it is clear that the 1-3 compound possesses a plane of symmetry, while the others are asymmetric. The objection to this point of view is that it is to be expected that there would exist more than one 1-3 compound and that these various compounds would show differences in crystalline and probably in chemical properties. No such differences have been observed. Until X-ray data are available as to the structure of some of these substances it is of little use pushing the question further.

Another interesting problem in crystallography is the determination of the conditions governing the choice by any molecule of some particular crystal form. If the molecule is asymmetric it is theoretically possible for it to crystallise in any of the 32 classes. On the other hand, it seems extremely improbable that 48 molecules will arrange themselves in such a way as to form a crystal of class 32. In fact, it appears that eight is about the maximum number of molecules probable in the elementary cell. It is to be expected that asymmetric molecules will crystallise
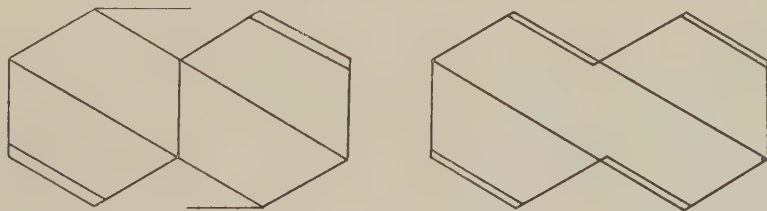
Fig. 14.

in classes of low symmetry, and this is amply supported by the measurements made by crystallographers. For example, a statistical investigation of about 1,000 of the aromatic organic crystals described by Groth shows that approximately 60 per cent. form crystals which would require four asymmetric molecules, about 20 per cent. would require eight, and about 20 per cent. four. Some of these molecules may have some symmetry of their own, and the 20 per cent. containing eight is probably an over estimate. It seems that there is a decided tendency for four asymmetric molecules to come together, grouping themselves about an axis and plane of symmetry.

If the molecule possesses symmetry and our second hypothesis is true, that the symmetry of the molecule is reproduced in the crystal, then it is clear that the selection will fall on more highly symmetrical classes. The symmetry of the molecule imposes a lower limit to the symmetry of the crystal. It appears to be generally true that there is more than one molecule in the elementary cell. This means that the crystal is of higher symmetry than the component molecule. The only cases where we should expect only one molecule in the cell are in classes 1 and 32, the latter with a molecule or atom which in itself possesses perfect symmetry if such exists.

While these conditions impose certain restrictions on the choice of a crystal form by any particular molecule they still leave a considerable margin of choice. The remaining conditions must depend on the molecule and it seems fairly certain that an important factor is the distribution of the residual forces round the molecule. That such forces exist is shown by the fact that the crystals actually form. These forces are presumably of an electrical nature and it is on the distribution of these charges over the molecule that the final choice of a crystal form must depend. It is possible to imagine distributions of positive and negative charges over the molecular surface such that when two molecules came within each other's sphere of action they would naturally orientate themselves about a digonal or other axis of symmetry.

In the case of simple polar compounds, the two atoms of the molecule are oppositely charged and there will be a strong tendency for the negatively charged atom of any molecule to become surrounded by the positively charged atoms of other atoms and *vice-versa*. This is what happens in the case of NaCl and similar molecules.

Some of the considerations stated above have recently been used in attempts to solve the structure of somewhat complicated crystals and have proved of considerable value in the interpretation of the data derived from X-ray observations.

SUMMARY.

1. The nature of the elementary lattice and the meaning of axes and planes of symmetry in the lattice are discussed.

2. A table is given showing the number of asymmetric molecules necessary per unit lattice to produce the symmetry of the various crystal classes. The existence of fewer molecules in the unit cell is shown to imply the existence of symmetry in the molecule.

3. The relative positions of the variously orientated molecules in the lattice are considered, and it is shown that only certain positions are possible.

4. It is suggested that the number of molecules in the elementary cell is always the minimum necessary to satisfy the symmetry conditions ; also that any symmetry of the molecule is reproduced in the crystal.

5. Applications of these principles to particular cases of inorganic and organic crystals are given.

6. The factors underlying the selection by a molecule of some particular crystal class are briefly discussed.

In conclusion, the author has much pleasure in acknowledging his indebtedness to Prof. Sir W. H. Bragg, F.R.S., for much help during the development of these ideas.

This Paper originated in connection with work being carried out at University College, London, under the Department of Scientific and Industrial Research.

DISCUSSION.

The PRESIDENT, before calling on Sir William Bragg, congratulated him in the name of the Society on the honour recently conferred upon him by the Paris Academy of Sciences.

Sir WILLIAM BRAGG, after acknowledging the congratulations tendered to him, said that the Paper before the meeting was an important one, throwing a new light on the possibilities of investigating crystal structure. The Author's intricate but lucid argument affords a great simplification of the problem. Three unitary structures have to be considered : (1) The chemical

molecule, as it exists in solution ; (2) the elementary crystal lattice or cell, consisting of groups of (3) crystal molecules, in which the atoms and electrons are not necessarily arranged in the same manner as in chemical molecules. The Author's method enables one to determine the symmetry of these groups at once from the geometry of the crystal, with a very little help from X-rays. A considerable vista of research is thus opened up, which may serve to settle such questions as that of the static versus the dynamic model of the atom.

Dr. J. W. EVANS thought that the methods employed by Mr. Shearer gave promise of proving of great assistance in working out the structure of crystals. There were, however, some considerations that must not be neglected :

(1) It is by no means certain that molecules always maintain their identity in the crystalline state.

(2) Even where they exist in the crystal, they may not retain the same form and symmetry as in a free state.

(3) Although in an ideal crystal all the cells are identical, and have identical orientation, this is by no means always actually true. We know that cells of different composition may be employed indiscriminately if they are approximately the same in form and volume, as in the case of the plagioclase felspars.

In the same way, if the outer forms of cells of the same substance approximate to a higher symmetry than the cell itself, these cells may, even when differently orientated, be employed in building up a crystal structure. The same may be true with cells that are enantiomorphically related. In these cases the whole structure may have a higher symmetry than the individual cells as the result of either (a) a symmetrical arrangement of the cells in the nature of molecular or ultramicroscopic twinning—twinning is usually an attempt to attain higher symmetry—thus forming lattice cells of greater size ; or (b) indiscriminate occurrence of cells differing in orientation or in enantiomorphic character. In the former case it might be very difficult to recognise by means of X-ray analysis the primary cells. In the latter case it would, I should think, be impossible.

The explanation of the crystalline structure of potassium chloride is not convincing. The symmetry of the molecules shown in Fig. 11 is that of the quartz or trigonal trapezohedral class which is represented by the symbol IIIUh (J. W. Evans, Min. Mag., Vol. 15, pp. 398-400, 1910), with three digonal axes at right angles to a trigonal axis, and making angles of 120° with each other ; whereas that of the crystal structure is stated to belong to the cuprite or pentagonal eikositetrahedral, CUh, class, in which there are four trigonal axes corresponding to the diagonals of the cube and three digonal axes at right angles to one another and parallel to the edges of the cube. The cell contains four molecules, and the trigonal axes of these are supposed by the author to be parallel to the diagonals of the cube, but it is not explained how the 4×3 digonal axes at right angles to the diagonals are transformed into three digonal axes at right angles to one another.

From the examination of the material prepared at the Imperial College and a comparison of the work of other observers, Miss Knaggs, of the Imperial and Birkbeck Colleges, has shown that the symmetry of the crystal structures of the simpler carbon compounds frequently bears close relation to that of the molecule. Substances of the type CX_4, where X is an element, are usually cubic. Those of the types CX_3Y are trigonal or hexagonal, unless X is hydrogen, when the symmetry is lower. Those of the form $C(CX_3)_4$ are as a rule cubic, and those with $C(CX_2Y)_4$ are tetragonal. In all cases the symmetry is apt to be that of a higher or different class than would have been expected, though one belonging to the same system. In the case of the $C(CX_2Y)_4$ compounds the symmetry of the molecule would be that of chalcopyrite or tetragonal scalenohedral, IVBk, class, whereas that of the crystal usually belongs to the IVBc, or in one case IVBu class. It can be shown that symmetry of this character would be obtained by appropriate molecular twinning. Some of the compounds considered are dimorphic, crystals with less symmetry forming at lower temperatures.

The Speaker threw out the suggestion that crystal structure might possibly be due to the repulsion of the electrons of different atoms combined with an attraction of the atoms as a whole. If it be supposed that, contrary to the usual view, there are six outer electrons in chlorine having their mean position arranged like the points of an octahedron, and that in potassium there are eight electrons arranged like the points of a cube, it will be evident that a position of stability would be obtained when the points of each chlorine octahedron were opposite the centres of the faces of the surrounding potassium cubes, and the points of each potassium cube were opposite the centres of faces of chlorine octahedra, and that the potassiums and chlorines would then be arranged alternately in a cubic lattice just as we know to be the case.

Mr. SHEARER, in reply, thanked Sir William Bragg for his remarks on the Paper   He agreed with Dr. Evans as to the possibility of a difference between the molecule in a crystal and the molecule in solution. The two were not necessarily identical, although there was evidence that in many cases there was a strong resemblance.

Dr. Evans appeared to doubt the possibility of the determination of the true unit cell. What was done by X-rays was to measure the distance between two successive *identical* planes ; in this way, the method of X-rays was much more powerful than any other method at present at the disposal of the crystallographer.

Dr Evans was apparently under a misapprehension as regards the author's suggestions as to the KCl molecule. Although the KCl molecule was given the symmetry of the quartz class, there were present in the unit cell four such molecules. By giving these the proper orientations in the cell it was possible to give the crystal the symmetry of the pentagonal icositetrahedral class. The trigonal axis of the molecule lay along—not perpendicular to—the diagonal of the cube. The three digonal axes were parallel to three diagonals of the faces of the cube. Thus, if 111 were proportional to the direction cosines of the cube diagonal, or trigonal axis, the direction cosines of the three digonal axes of the molecule were proportional to $1\bar{1}0$, $\bar{1}01$ and $01\bar{1}$. These were parallel to three of the diagonals of the faces of the cube, and made angles of 120° with each other and were perpendicular to the trigonal axis. The directions of these axes coincided therefore with those of the cubic class to which KCl belonged. The rest of the symmetry of the crystal arose from the presence of four of these molecules in the unit cell.

The results obtained by Miss Knaggs were interesting, and appeared to support the suggestions put forward in the Paper. The crystal in general possessed more symmetry than the molecule owing to the presence of more than one molecule in the unit cell. Until the number of molecules in the cell had been determined it was unsatisfactory to discuss the symmetry of the molecule.

As far as external evidence went there seemed nothing in support of the suggestions of Dr. Evans as to the electron arrangements in the K and Cl atoms. Our knowledge of atomic and molecular forces was at present extremely limited.

X. *Modification of the Powder Method of Determining the Structure of Metal Crystals. By* E. A. OWEN, *M.A., D.Sc., and* G. D. PRESTON, *B.A., The National Physical Laboratory.*

ABSTRACT.

Plates of aluminium, iron, copper, lead and magnesium have been examined by means of the Bragg X-ray spectrometer employing radiation direct from a molybdenum anti-cathode. The maxima observed in the spectra are sufficiently intense to measure with accuracy, and the crystalline structure of the materials examined are readily determined. A few of the results obtained by the method are included in the Paper as typical examples.

I. INTRODUCTORY.

THE powder method of determining the structure of crystals, independently developed by Hull[*] and by Debye and Scherrer,[†] has had a wide application and has been extensively employed by Hull in particular for the examination of the structure of metal crystals. For this method the material under examination is reduced to a fine powder, a small quantity of which is placed in a tiny glass tube situated in the path of a beam of X-rays which has been rendered monochromatic by passing it through a suitable filter. The rays reflected at the different faces of the small crystals in the tube appear on a photographic film upon which they fall as a series of irregularly spaced lines, the positions of which suffice to solve the structure of the material.

W. H. Bragg[‡] has recently shown that the structure of powdered materials can also be solved by the aid of the X-ray spectrometer. The powder in this case is pasted on a flat sheet of cardboard and the whole mounted on the table of the spectrometer in the position usually occupied by the face of a single crystal. Un-filtered radiation direct from a rhodium anticathode falls on the powder, both the α and the β lines of the K series spectrum of rhodium being reflected in turn into the ionisation chamber. If the structure is not too complicated, the presence of the two lines is an advantage rather than a disadvantage in that the reflection of the one line serves as a check upon the other. The advantage of this method over the photographic method is twofold—the time of exposure is very much shortened and the relative intensity of the lines in the spectrum is directly observed. In order to obtain a readable impression of the spectrum upon a photographic plate an exposure of about 300 milliampere-hours is required, whereas with the X-ray spectrometer the spectrum can be plotted in an hour or two, using only about 1 milliampere through the bulb.

The present Paper is an account of a further development of the method of determining the structure of metal crystals. Instead of reducing the metal, the crystalline structure of which is required, to a fine powder, a plate of the metal is taken and examined by means of the X-ray spectrometer in a manner similar to that adopted by Bragg. The few results given below show that the structure can be solved by this means in a comparatively short time with only small currents passing through

* Hull, Phys. Rev., 10, p. 661 (1917).

† Debye and Scherrer, Phys. Zeitschr., 17, p. 277 (1916) ; 18, p. 291 (1917).

‡ W. H. Bragg, Proc. Phys. Soc., Vol. 33. Part 4, p. 222 (1921).

the bulb.   The examples chosen serve to illustrate the application of the method to
the solution of the structure of crystals belonging respectively to cubic and hexagonal
systems.

II.   Method of Procedure.

A plate of metal, either cast or rolled and measuring about 5 cm. by 2·5 cm.,
is subjected to heat treatment necessary to produce throughout its volume a mass
of small crystals—crystals about 1 mm. or less across are suitable for the purpose.
The plate is etched to remove the deformed surface layer of crystals or any oxide
that may have formed, and is then smeared over with vaseline to prevent further
oxidation.   It is mounted on the spectrometer with one of its large surfaces adjusted
so as to fall just over the centre of the crystal table.

The spectrometer is provided with an 8 in. circle graduated to read directly to
5 seconds of arc.   The width of the chamber slit is kept at 2 mm., which corresponds
to about one degree of arc, whilst the bulb and crystal slits are opened wide enough
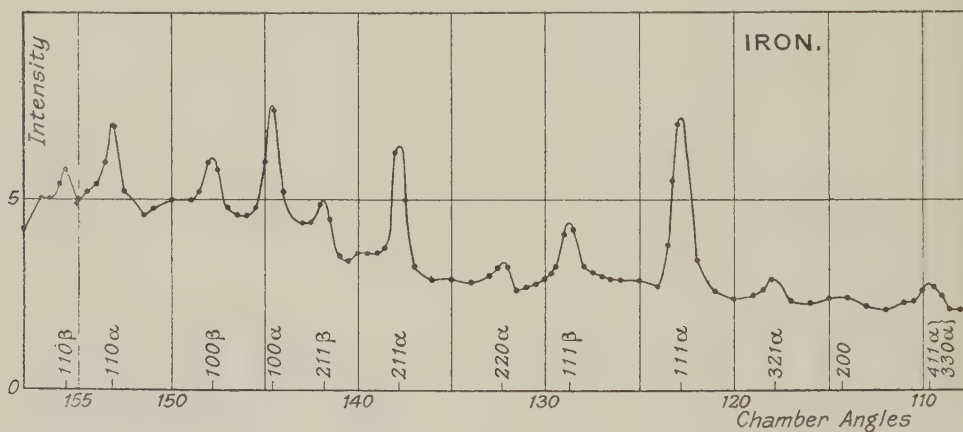
Fig. 1.

to allow a beam of rays to fall upon the plate just sufficient to cover it from side to
side.

During a few preliminary observations the metal plate is fixed at any angle
with the X-ray beam and readings are taken of the ionisation for different positions
of the chamber.   One or two of the main maxima are located in this way.   The
chamber is then fixed in the position to receive one of these maxima, whilst the plate
is rotated through a range of angle and the position of the plate determined at which
maximum ionisation occurs in the chamber.   Having determined this position, the
plate is set at that angle and both chamber and crystal table are rotated together,
the former at twice the rate of the latter.

These preliminary observations determine the position of the plate at which
there is the smallest amount of absorption of the radiation for the face corresponding
to the particular maximum chosen.   We are assuming that this condition obtains
also approximately for all the other faces of the crystals, since the number of crystals
in the area of the plate irradiated is very large and there will be as many crystals on
the average whose 100 planes, for example, are in the surface of the plate as there

are crystals presenting 111, 110 or any other plane. That this assumption is justi-
fied is proved by the relative intensity of a line reflected at a large angle when the
plate is kept fixed and when it is rotated in the manner stated, the intensity in the
latter case being found to be much greater than that in the former. In the present
investigation no particular attention has been directed to the relative intensities
of the lines in the spectra, the main object being to locate the lines over a wide range
of crystal planes.

III. Results.

The five metals, magnesium, aluminium, iron, copper and lead, have been
examined in detail. The curve obtained by the above method for iron is shown
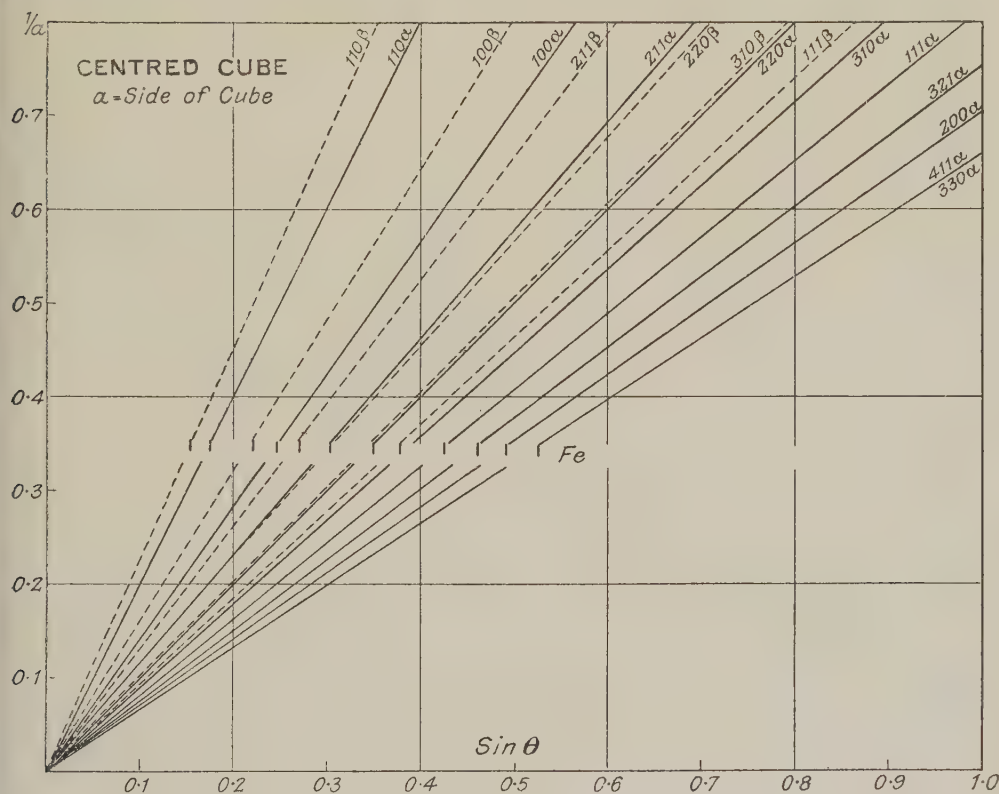
FIG. 2.

in Fig. 1, the radiation employed being that from a molybdenum anticathode for
which $\lambda_\alpha = 0.711\text{Å}$ and $\lambda_\beta = 0.629\text{Å}$. The curves show the actual readings of
intensity plotted against chamber angle, readings being taken at half degree intervals.
The exposure for each reading was about 8 seconds. The Coolidge bulb employed
was operated at 80,000 volts and had a current of 1 milliampere passing through it.

When reflection takes place at a crystal face the relation $n\lambda = 2d \sin \theta$
must be satisfied, where λ is the wave-length of the radiation reflected, d the

distance between successive planes parallel to the face, and θ the angle of reflection. In the case of crystals of cubic structure, d for all faces is proportional to the length of the side of the elementary cube. Thus we have $n\lambda = 2ka \cdot \sin\theta$, where k is a constant characteristic of the set of planes at which reflection occurs. The values of k for different sets of planes in any one of the three possible cubic structures are fixed so that if we plot the reciprocal of a against $\sin\theta$ we get a series of straight lines passing through the origin, each straight line corresponding to one definite face of the crystal. The graphs for the face centred cube and for the centred cube
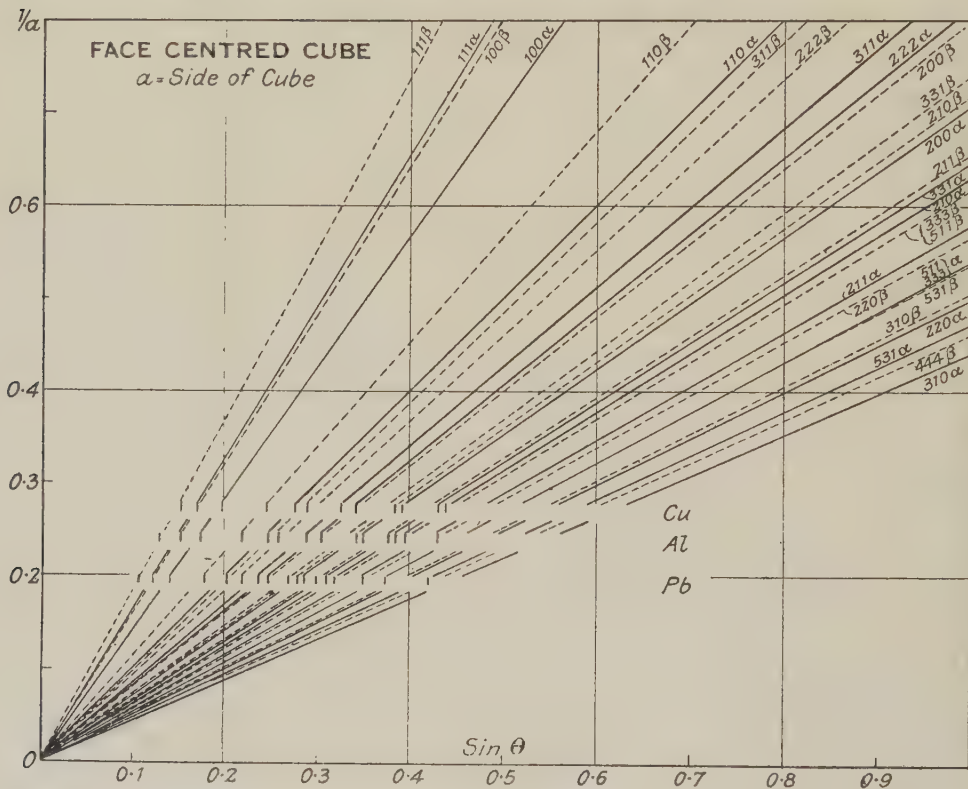
FIG. 3.

are shown in Figs. 2 and 3 respectively. In these graphs both the α and the β lines are included, the former being the full and the latter the broken lines. If the structure of the crystal is cubic, the lines in its spectrum should fit somewhere on one of these graphs. To test this the sines of half the chamber angles of the lines in the spectrum were set out on a slip of paper, which was passed over the graphs, the zero line of the one coinciding with the zero line of the other. We have shown in Figs. 2 and 3 where the best fit occurs in each case. Aluminium, copper and lead fit on the graph for the face centred cube and iron on that for the centred cube. The magnesium spectrum does not fit on any of these graphs and will have

to be considered separately. In the case of each of the other four metals the fit is good and, with one or two exceptions, *all* the lines are accounted for.

From the position at which each spectrum fits the graph, we deduce at once the length of the side of the elementary cube. To obtain a more accurate determination of this quantity, its value was calculated independently for each line in the spectrum. The values calculated for iron are tabulated below and serve as an example of the order of agreement obtained.

TABLE I.—*Electrolytic Iron.*

Reading of Maximum.	θ	sin θ	Form.	$k = d/a$	Side of Elementary Cube. (a)
155·6°	8·85	0·1539	110β	0·707	2·867$\overset{\circ}{A}$
153·1°	10·1	0·1754	110α	0·707	2·867
147·8°	12·75	0·2207	100β	0·500	2·828
144·6°	14·35	0·2478	100α	0·500	2·869
141·8°	15·75	0·2714	211β	0·408	2·870
137·8°	17·75	0·3048	211α	0·408	2·909
132·3°	20·5	0·3502	220α	0·3535	2·872
128·7°	22·3	0·3795	111β	0·2885	2·849
122·8°	25·25	0·4266	111α	0·2885	2·888
117·9°	27·7	0·4648	321α	0·2672	2·863
114·2°	29·55	0·4932	200α	0·250	2·882
109·7°	31·8	0·5270	(411α) (330α)	0·2358	2·863
				Mean value of $a = 2\cdot869\overset{\circ}{A}$.	

The results obtained with the four metals aluminium, iron, copper and lead are summarised in Table II. Included in the table are the values of the number of atoms (n) associated with each elementary cube calculated from the observed mean values of a. In the last column are inserted for comparison the values of a found by Hull with the powder method. The greatest difference between these values and those obtained by the method described here is about 1 per cent. The calculated values of n in the next to the last column of the table are in accordance with the lattice assigned to the different elements, namely that for a centred cube, two atoms, and for a face centred cube, four atoms, are associated with each elementary cube.

TABLE II.

Element.	Structure.	Length of side of elementary cube (a).	$n = \dfrac{\rho a^3}{m}$	Values of a as measured by Hull.
Al	Face Centred Cube	4·041$\overset{\circ}{A}$	3·98	4·05A
Fe	Centred Cube	2·869	2·015	2·86
Cu	Face Centred Cube	3·628	4·06	3·60
Pb	Face Centred Cube	4·983	4·11	4·92

IV. MAGNESIUM.

The procedure in the case of magnesium had to be somewhat modified in order to include crystals of different axial ratios. Considering, for instance, crystals of the close packed hexagonal type, a graph was drawn for each set of planes in the crystals giving the relation between the logarithm of the reciprocal of the axial ratio (c) and the logarithm of the reciprocal of the distance between consecutive planes, assuming the side of the hexagon to be unity. Such a graph will be independent of the wave-length of the radiation reflected, and can, therefore, be used both for
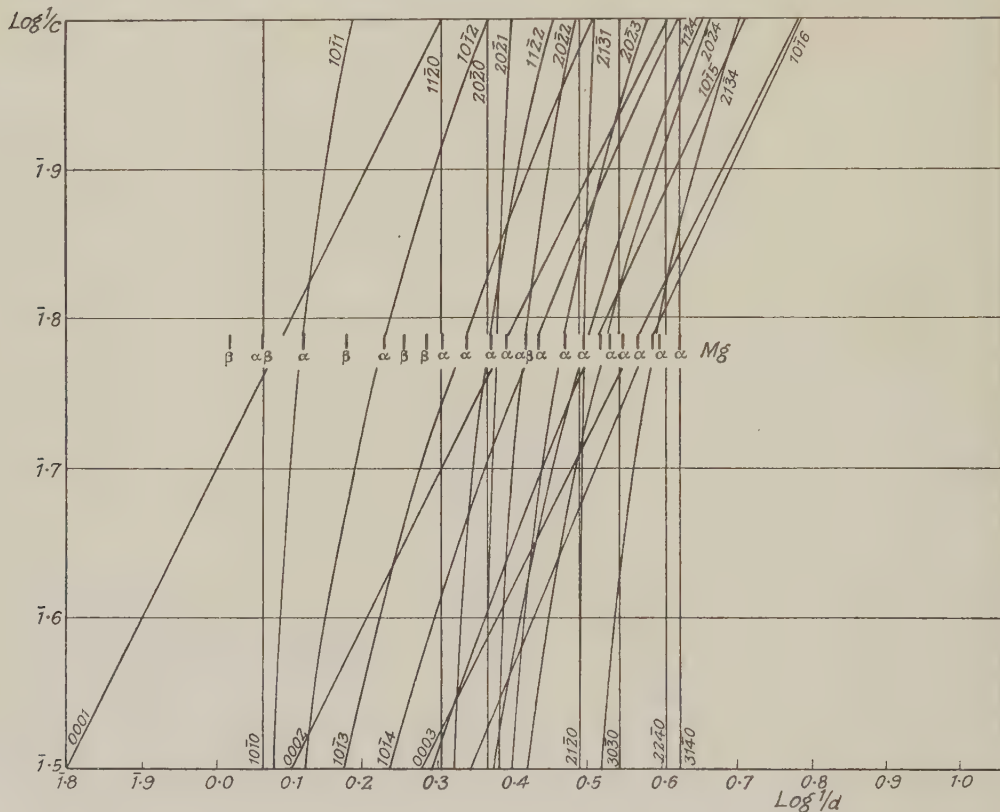
FIG. 4.

the α and for the β lines. The graph, shown in Fig. 4, is that for a closed packed hexagonal lattice. It can be used for crystals over a wide range of axial ratio.*

It should be mentioned that the magnesium plate examined was cut from a rod of the metal about 2·5 cm. wide. The specimen was polished and etched before the X-ray examination was conducted, but in this case it was not especially heat treated previously as in the other cases already considered.

The lines observed in the spectrum of this metal were very prominent, and as many as 22 lines were measured, including both α and β lines. The logarithm of the

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* A graph similar to this has been employed by Hull, the only difference being in the choice of the ordinates (*see* Hull, Phys. Rev., 17, p. 549, 1921).

TABLE III.—*Magnesium.*

Reading of maximum.	θ	Log. sin θ	Form (Miller-Bravais).	d	d/a (c=1·631)	Length of side of hexagon. (a)
160·2°	6·65	$\bar{1}\cdot 0633$	$10\bar{1}0\ \beta$	$a\dfrac{\sqrt{3}}{2}$	0·866	$3\cdot 116\overset{\circ}{A}$
158·5°	7·4	·1099	$10\bar{1}0\ \alpha$	$a\dfrac{\sqrt{3}}{2}$	0·866	3·187
			$10\bar{1}1\ \beta$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{1}{c^2}}$	0·7649	3·162
156·4°	8·45	·1673	$10\bar{1}1\ \alpha$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{1}{c^2}}$	0·7649	3·161
154·1°	9·6	·2222	$10\bar{1}2\ \beta$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{4}{c^2}}$	0·5937	3·151
151·7°	10·8	·2727	$10\bar{1}2\ \alpha$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{4}{c^2}}$	0·5937	3·186
150·4°	11·45	·2980	$11\bar{2}0\ \beta$	$a/2$	0·500	3·142
148·5°	12·4	·3318	$10\bar{1}3\ \beta$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{9}{c^2}}$	0·4605	3·156
147·4°	12·95	·3506	$11\bar{2}0\ \alpha$	$a/2$	0·500	3·171
145·2°	14·05	·3850	$10\bar{1}3\ \alpha$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{9}{c^2}}$	0·4605	3·181
142·7°	15·3	·4215	$11\bar{2}2\ \alpha$	$a\Big/\sqrt{4+\dfrac{4}{c^2}}$	0·4263	3·159
141·5°	15·9	·4378	$0002\ \alpha$	$ac/4$	0·4077	3·181
139·3°	17·0	$\bar{1}\cdot 4660$	$20\bar{2}2\ \alpha$	$a\Big/2\sqrt{\dfrac{4}{3}+\dfrac{1}{c^2}}$	0·3824	3·178
			$20\bar{2}3\ \beta$	$a\Big/\sqrt{\dfrac{16}{3}+\dfrac{9}{c^2}}$	0·3388	3·150
137·8°	17·75	·4840	$10\bar{1}4\ \alpha$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{16}{c^2}}$	0·3689	3·161
			$21\bar{3}1\ \beta$	$a\Big/\sqrt{\dfrac{28}{3}+\dfrac{1}{c^2}}$	0·3209	3·190
134·8°	19·25	·5181	$20\bar{2}3\ \alpha$	$a\Big/\sqrt{\dfrac{16}{3}+\dfrac{9}{c^2}}$	0·3388	3·183
132·5°	20·4	·5424	$21\bar{3}1\ \alpha$	$a\Big/\sqrt{\dfrac{28}{3}+\dfrac{1}{c^2}}$	0·3209	3·177
130·0°	21·65	·5669	$10\bar{1}5\ \alpha$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{25}{c^2}}$	0·3053	3·156
128·7°	22·3	·5794	$20\bar{2}4\ \alpha$	$a\Big/2\sqrt{\dfrac{4}{3}+\dfrac{4}{c^2}}$	0·2968	3·154
127·0°	23·15	·5945	$30\bar{3}0\ \alpha$	$a\sqrt{3}/6$	0·2886	3·133
124·9°	24·2	·6127	$0003\ \alpha$	$ac/6$	0·2718	3·190
122·7°	25·3	·6308	$10\bar{1}6\ \alpha$	$a\Big/\sqrt{\dfrac{4}{3}+\dfrac{36}{c^2}}$	0·2593	3·207
121·5°	25·9	·6403	$21\bar{3}4\ \alpha$	$a\Big/\sqrt{\dfrac{28}{3}+\dfrac{16}{c^2}}$	0·2553	3·188
117·5°	27·9	·6701	$31\bar{4}0\ \alpha$	$a\sqrt{\dfrac{3}{52}}$	0·2402	3·168

Mean value of $a = 3\cdot 17\overset{\circ}{A}$

sine of the angle of reflection of each line in the spectrum was plotted on a slip of paper, and passed over the graph (Fig. 4) as previously until a position was found at which the lines fitted. In this case it is difficult to decide which are the α and which are the β line reflections. In order to ascertain this, a filter of zirconium was placed in front of the chamber slit, and a small portion of the spectrum again examined. In this way the β lines were almost entirely cut out, leaving only the α lines.

Of the 22 lines in the spectrum, 18 were found to be α lines and four β lines. The value of the axial ratio taken was that obtained from the α lines alone, and was found to be 1·631. The length of the side of the hexagon was calculated for each set of planes in the crystal both for the α and for the β lines. These values are included in Table III., the mean value being $3 \cdot 17 \, \overset{\circ}{A}$.

As a further test on the structure, we may calculate the number of atoms associated with each elementary cell. The volume of the elementary triangular prism is $\dfrac{\sqrt{3}}{4} a^3 c$, where a is the side of the hexagon and c the axial ratio. Then n, the number of atoms associated with each elementary triangular prism $= \dfrac{\sqrt{3}}{4} \cdot \dfrac{a^3 c \, \rho}{m}$, where m is the mass of the atom and ρ the density of the material.

Hence,
$$n = \frac{\sqrt{3}}{4} \times \frac{(3 \cdot 17)^3 \times 1 \cdot 631 \times 1 \cdot 74}{24 \cdot 13 \times 1 \cdot 663} = 0 \cdot 98.$$

There is, therefore, one atom associated with each elementary cell, which conforms to the requirements of a hexagonal lattice of the close packed type.

A point which needs further explanation is the absence of the 0001 line in the observed spectrum curve. This fact was also noted by Hull, but on fuller investigation of a number of single crystals he found the line to be present. In the present instance the region of the spectrum where the line should appear was examined in greater detail, observations being taken at 10′ intervals instead of 30′ intervals. In this case the line was clearly indicated, although its intensity was small.

The above measurements suffice to show the utility of the method. It promises to be a valuable asset to the metallurgist and has an advantage over the powder method in that it allows the specimen, after being polished and etched for microscopical examination, to be used directly on the X-ray spectrometer. In addition, there is in this case no possible distortion of the crystals with consequent increased absorption of the radiation.

We desire to express our indebtedness to the Metallurgy Department of the Laboratory for their assistance in the preparation of the specimens.

DISCUSSION.

Sir WILLIAM BRAGG congratulated the Authors on their work, which indicates that the technique of crystal analysis is improving and that its methods are coming into general use. A consideration to be borne in mind is that one of the crystal planes may become unduly accentuated by the polishing process. Further, in the case of aluminium and other metals it is possible for individual crystals to be very large.

Dr. G. D. WEST, after dwelling on the advantages of focussing the X-ray beam by inclining the plane in which the crystals lie, inquired as to the width of the slits employed as the conditions varied, and as to the allowance to be made for variations in the output of the bulb.

Mr. PRESTON, in reply, stated that the Authors had been careful to use crystals less than 1 mm. in size. The chamber slit was kept at 2 mm., and the bulb slit varied from 2 mm. to 4 mm. for different parts of the spectrum. The output of the bulb could be kept constant by regulating the filament current, but for moderate accuracy little regulation was required.

XI. *The Cathode Ray Oscillograph.* By A. B. Wood, D.Sc., F.Inst.P.

ABSTRACT.

The Paper deals with a new form of cathode-ray oscillograph adapted for commercial production and laboratory use. The instrument described is of the low-voltage type, in which a hot cathode is employed as a source of the electron current. This low-voltage type of oscillograph is much more sensitive than the high-voltage cold cathode type designed by M. Dufour. Various methods are described for focusing the cathode-ray stream, and a proposal has been made for an oscillograph with external (*i.e.,* outside the vacuum) photographic film. Experiments have been made to determine the most suitable photographic film or plate. Ordinary gelatine coated roll films or plates are unsuitable, owing to the marked absorption of the cathode-rays by the gelatine. The best results have been obtained with Schumann plates containing a proportion of calcium tungstate. This material phosphoresces with a light rich in ultra-violet, and consequently the secondary luminous effect on the Schumann plate is very great.

Mechanical, electrostatic and electromagnetic methods are described for generating a time-axis on the records. For certain purposes this time axis is sinusoidal, whilst for others it is linear. Numerous records of high-frequency A.C. wave forms and of impulsive electrical phenomena have been obtained, and a few of these are reproduced in the Paper. Brief reference is made in conclusion to the applications of the oscillograph to research and electrical engineering problems, where other well-known forms of oscillograph (Duddell, Einthoven, &c., types) cannot be employed on account of the inertia of the moving element.

I. Historical.

With the development of wireless telegraphy and telephony in which electrical oscillations of very high frequency are in everyday use, the need for a sensitive and efficient high-frequency oscillograph is becoming more acute. In the study of impulsive electrical phenomena also the need for such an oscillograph has long been felt.

Oscillographs at present in use, *e.g.,* the Duddell and the Einthoven types, although excellent at comparatively low frequencies, fail to give reliable records at frequencies much in excess of 1,000 $\sim$ per second, for the true wave-form of the oscillation to be recorded is masked by the natural vibration inherent in the instrument itself. The upper limit of frequency which such an oscillograph can record faithfully is determined by the mechanical inertia and elastic limits of the moving parts and controls. At very high frequencies these oscillographs become very insensitive.

The first serious attempt to design an oscillograph in which the moving element was so light that it could respond to extremely high frequencies is due to Braun. He made use of the facts that a pencil of cathode-rays is easily deflected by magnetic and electrostatic fields, and that the rays produce visible and photographic actions under suitable conditions. It is well known that the mechanical inertia of the cathode rays is quite negligible even at extremely high frequencies (of the order 10^6 per second). The Braun tube has been described in many text-books of electricity, and requires little more than a brief reference here. In principle it consists of a vacuum tube in which a stream of cathode rays is produced by means of a H.T. supply of about 50,000 volts. Some of these cathode rays pass through a fine tube or pinhole in the anode and thence between the poles of a magnet and a pair of

electrostatic deflecting plates. Variations of current in the magnet (or of voltage on the deflecting plates) produce corresponding deflections of the pencil of cathode rays, which ultimately falls on a screen of phosphorescent material and is thus rendered visible. The sensitivity of such an oscillograph is constant at *all* frequencies. Attempts to photograph wave-forms of A.C. by means of this oscillograph were unsuccessful until an automatic method was devised whereby the cathode rays were constrained to retraverse exactly the same track across the photographic plate any number of times in succession. In this way records of high frequency wave-forms were obtained by " multiple-exposure " of the plate to the cathode rays—such a photograph requiring about 100 traverses of the rays across the plate to be satisfactory.

Now when very high frequency or impulsive phenomena of short duration are to be studied, it is always a matter of great difficulty to obtain satisfactory results with the multiple-exposure method ; in innumerable cases it is quite impossible to arrange for regular periodic repetition of the phenomena to be recorded. In such cases it is essential that a record should be obtained by a single traverse of the cathode rays across the plate.

An oscillograph which achieves this condition has recently been designed and constructed by M. Dufour. It is essentially an improved form of Braun tube, in which the cathode stream is much more intense than previously obtained. As in the case of the Braun tube, a cold cathode is employed and a trace of gas is left in the tube. The rays are produced by applying a p.d. of about 60,000 volts between cathode and anode. With this oscillograph Dufour has obtained beautiful records of frequencies of the order $10^5 \sim$ /sec. In this respect it is a great advance on the original Braun tube, but it has one serious drawback, viz., its sensitivity is low. Cathode rays produced under a p.d. of 60,000 volts attain a very high velocity. High velocity cathode rays are more difficult to deflect than low-velocity rays, the deflection being inversely proportional to the voltage of the rays. Hence in Dufour's oscillograph large electrostatic voltages and magnetic fields are required to produce satisfactory deflections on the oscillograph records. On the other hand, it must be remembered that the photographic effect of high-velocity rays is greater than that of rays of low velocity. The experimental difficulties increase therefore as the velocity of the cathode rays diminishes.

In connection with certain experiments a sensitive cathode ray oscillograph has been employed by D. A. Keys, who was experimenting on a method proposed by Sir J. J. Thomson.* For these experiments it was essential that the oscillograph should have considerably greater sensitivity than that used by Dufour, for the voltage changes to be recorded were comparatively small. The increased sensitivity was obtained by using a p.d. of about 5,000 volts only across the tube, whilst the cathode was a tungsten filament heated by means of an insulated battery of accumulators. The use of a hot filament gives a powerful steady supply of electrons for the cathode stream, sufficient to produce the required photographic effects. With this apparatus Keys obtained a number of records of underwater and gaseous explosions, but the apparatus was rather fickle in its behaviour and left much to be desired as regards mechanical construction and adjustability.

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* Sir J. J. Thomson, Engineering, Vol. 107, 1919, pp. 543-544 ; and D. A. Keys, Phil. Mag., 42, pp. 473-488, Oct. (1921).

II. Development of Low-voltage Type of Oscillograph.

Cathode ray oscillographs may be divided broadly into two classes, according as they employ :—

(*a*) A cold cathode—as in the Braun tube and Dufour oscillograph. These are of the high-voltage type (of the order 10,000 volts), are not electrically sensitive, but are very efficient photographically.

(*b*) A hot cathode—as in the oscillograph used by D. A. Keys. The voltages used in this case to produce cathode rays may vary according to requirements, and may range from 100 volts to 10,000 volts. Such oscillographs are of the low-voltage type, are very sensitive electrically, but are not sensitive photographically.

In what follows we are concerned with (*b*) the low-voltage type of oscillograph
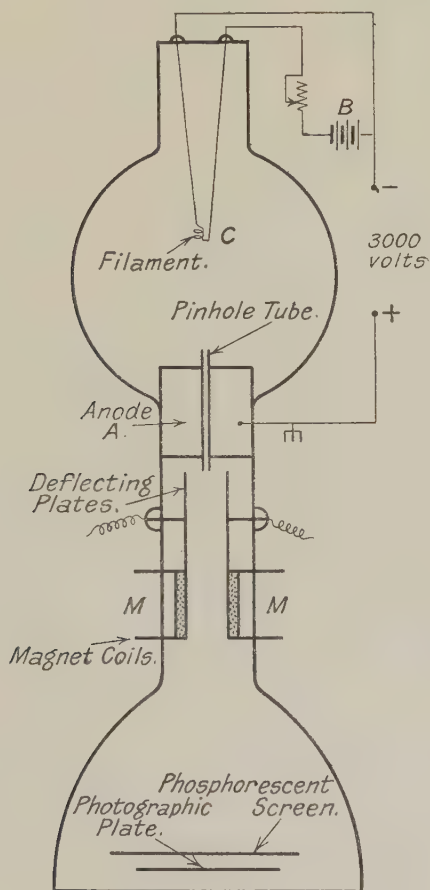
Fig. 1.—Oscillograph Connections.

employing a hot cathode. The general arrangements of the electrical circuits of such an oscillograph are indicated diagrammatically in Fig. 1. The cathode C consists of a spiral of thin tungsten or lime-coated platinum wire, which is heated

by means of an insulated battery B. The hot spiral is supported inside a cylindrical sheath (to which one end is connected) which exerts a focusing action on the cathode

Ebonite Cap.

Glass Bulb.

Spiral Filament.
Tungsten or Platinum
(Lime Coated)

Pinhole Tube.

Drying Bulb.

Brass Body.

Pumping Tube.

Glass

E.S. Deflecting Plates.
(Attached to, but
Insulated from Pole-
pieces of Magnet.)

Electrodes of
Deflecting Plates.

Electro-magnet
(Two Windings)

Plate Glass (1" thick)

Camera
Chamber.

Phosphorescent
Screen

Camera
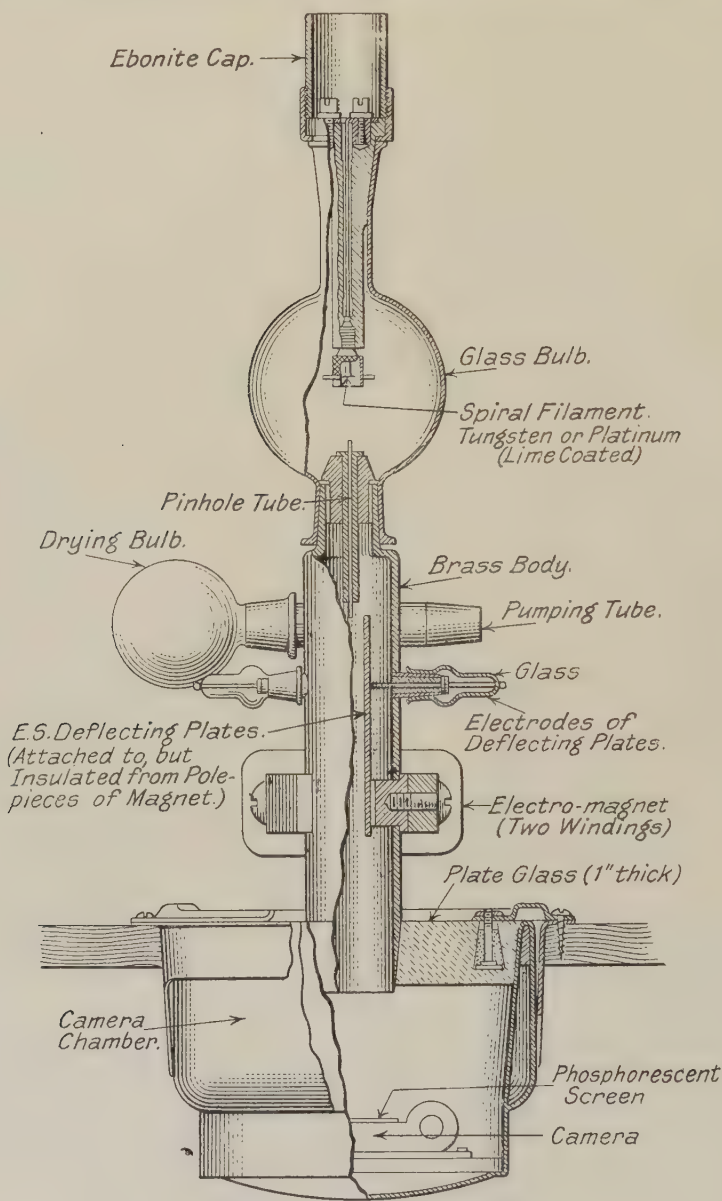
FIG. 2.—CATHODE-RAY OSCILLOGRAPH.

rays as they travel towards the anode A. The rays are accelerated by means of an applied p.d. of, say, 3,000 volts, the positive end being connected to the anode A

and earthed. After passing through the pinhole tube in the anode the cathode rays travel through parallel electrostatic and magnetic fields and ultimately fall on the phosphorescent screen. Removal of this screen exposes a photographic plate to the action of the rays. The lower the voltage at which it is required to generate the cathode stream the nearer together should be the cathode and anode. With rays of lower voltage than 1,000, however, considerable difficulty is experienced in obtaining a sufficiently intense beam to give a photographic effect.

Various methods may be employed in generating the time axis when an oscillograph record is to be made. In practice it is usually found inconvenient to employ actual mechanical movement of the photographic film, the same end being
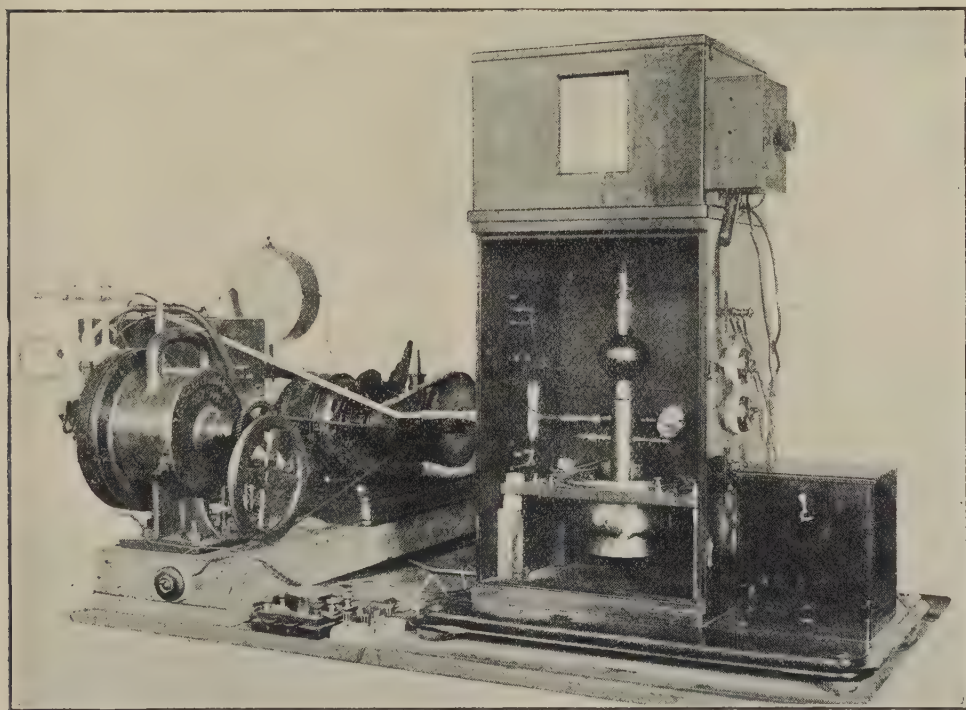
Fig. 3

achieved much more conveniently by sweeping the cathode stream across the plate in a known manner either by electrostatic or electromagnetic means.

On account of the extended applications of the above type of low-voltage oscillograph it was considered advisable to design a simple and robust form which might be of general service as a laboratory instrument. Fig. 2 indicates broadly the main features of a design proposed to fulfil such conditions. As far as possible fragile glass-work has been eliminated, the only glass parts liable to fracture being the main bulb carrying the heated cathode drying bulbs and a pair of small side connecting tubes. It will be observed that the tube containing the E.S. deflecting

plates is made entirely of brass. The various parts of the apparatus are fitted together and rendered airtight by means of accurately fitting greased conical joints. The oscillograph is thus very easily assembled or taken apart, and spare fittings can be kept in reserve to replace quickly in case of accidental damage to any part of the apparatus. The oscillograph when assembled in this manner is found to be perfectly airtight, an X-ray vacuum being attained after about 10 minutes pumping with the standard Gaede and auxiliary vacuum pumps.

A photograph of the oscillograph mounted in its cabinet is shown in Fig. 3. The filament battery, control rheostat, ammeter, voltmeter and other fittings connected to the cathode and consequently at high potential (possibly up to 5,000 volts) are collected in the upper part of the cabinet, above the cathode ray tube itself, and insulated on an ebonite base. The high-tension supply employed in the experiments to be described is obtained from a 5,000 volt generator made by Evershed & Vignoles. The generator is provided with an automatic "cut-out," which prevents the space-current in the tube from rising above 20 milliamps. The vacuum conditions in the apparatus are examined qualitatively by means of an auxiliary test-bulb connected to the secondary of a spark coil.

III. Details of Design.

It may be of interest to consider some of the more important details of design of the oscillograph.

(a) The Cathode.

Good results have been attained by using filaments of either tungsten or limecoated platinum. The latter have in general been found more satisfactory, for

(a)   (b)   (c)   (d)   (e)
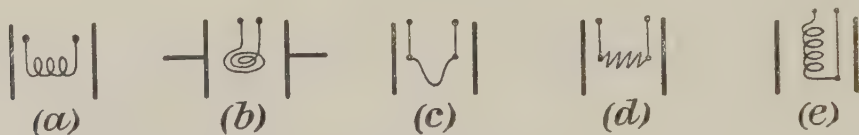
Fig. 4.—Typical Filaments.

they emit a very powerful electron stream at a comparatively low temperature, and there are as a consequence fewer casualties owing to "burning out" filaments. A simple and efficient method of lime-coating the platinum filaments is to dip them in a solution of calcium nitrate,* then heat electrically to convert this into oxide. A filament once coated in this way will serve its purpose efficiently for several weeks constant use without recoating.

The form of the filament is a matter of some considerable interest both from a physical and mathematical standpoint. The ultimate direction in which the cathode rays leave the filament depends primarily of course on the magnetic and electrostatic fields at the surface of the filament. These fields must have a very powerful initial effect, and as a consequence it becomes a matter of some importance to determine the best shape of the filament. The mathematical aspect of the question

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* The author has recently been informed that this method is improved by adding a small quantity of the salt of a noble metal to the solution—the oxide layer then sticks more tenaciously to the platinum filament.

is under consideration, and it is hoped that we shall soon be able to design a filament which will bring a wide beam of cathode rays to a sharp focus. Such a concentrated pencil of rays would be ideal for oscillographic purposes. For the present, however, the design of the filament is based mainly on experimental evidence. Some of the many forms of filament used are shown diagrammatically in Fig. 4. Of these the flat spiral filament shown in Fig. 4 (b) has given the best results. It has been found a definite advantage to employ a cylindrical sheath shrouding the filament as shown.

(a) Heating Coil

Aux^y Cathode heated by bombardment

(b) Mechanical

Focussing Tubes.

(e)

(c) Electro-Magnetic.

Coil to prevent beam spreading.

(d) Electrostatic

P.D. between central wire and outer tube (wire slightly +ve)

E.S. Attraction towards wire results in a focussing effect.
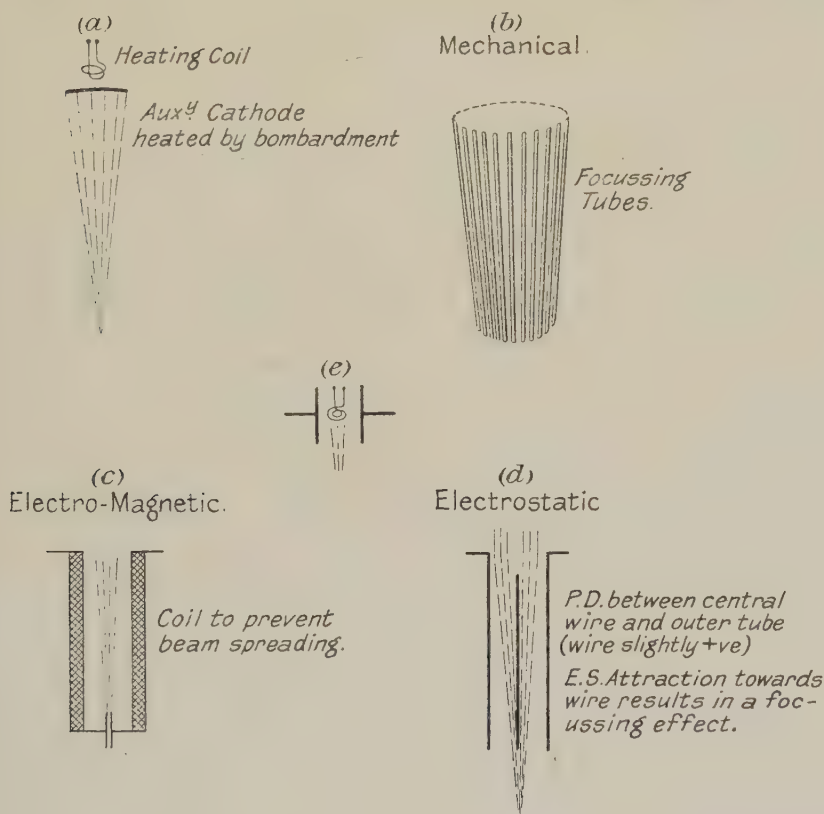
FIG. 5.—TYPICAL METHODS OF FOCUSING A WIDE BEAM OF CATHODE RAYS.

The flanged sheath (as in the Coolidge tube) has been used, but it is not possible to state that it is definitely better than the unflanged form.

(b) Focusing Devices.

One of the chief difficulties encountered in the design of a low-voltage oscillograph is that of obtaining a sufficiently intense pencil of rays to affect the photographic plate when moving over it at high speeds. Two methods at once present themselves, (1) by increasing the intensity of the cathode ray stream, and (2) by increasing the sensitiveness of the photographic plate. We shall for the

present discuss the first of these methods, leaving the question of photography till later.

The intensity of the cathode stream can be increased up to a certain point by increasing the current through the filament, but an upper limit in this direction is soon reached at a temperature just below the fusing point of the filament. Clearly, if very great advances are to be made, they must be in directions other than this. One alternative is to increase the intensity of the beam by focusing the rays from an extended area of filament. Typical methods of doing this are indicated in Fig. 5. These methods may be broadly divided into :—

(i) *Mechanical*—by means of concave cathodes, or by the use of cones of fine tubes, &c., as in Fig. 5(*a*) and (*b*).

(ii) *Electro-magnetic*—by means of a solenoid carrying a current and surrounding the anode and Fig. 5(*c*). Strictly speaking, this method is " anti-spread ing " or " paralleling " rather than actual focusing.

(iii) *Electro-static*—by means of a positively charged central wire through the anode (*see* Fig. 5(*d*)). The wire attracts the negatively-charged rays, causing the beam to converge towards the further tip of the wire. Another method, shown in Fig. 5(*e*), is that employed in Coolidge X-ray tubes, where the filament sheath repels the electron stream towards the axis.

The mechanical method (i) shown in Fig. 5(*b*) whilst giving some evidence of focus-ing action did not result in any material improvement of the intensity of the cathode ray pencil. Methods (ii) and (iii) are considerably more promising, but as yet the experiments are not completed. The sheathed filament of the flat spiral form, shown in Fig. 4(*b*), has for the present been adopted, as it gives more satisfactory results than any others yet examined.

It is still a moot point how far any form of focusing device is advantageous, for the beam of rays streaming from a hot cathode always appears as a heterogeneous bundle of fine pencils of rays varying between wide limits of intensity. When it is possible to deflect one of the more intense of these pencils into and through the pinhole tube a brilliant small recording spot is obtained which is usually quite satisfactory for high speed oscillograph work. Of course, focusing devices will increase the probability of such an intense fine pencil entering the pinhole tube, but it is not at present certain that the intensity of the recording spot is greatly increased by the focusing action.

Another method of heating the cathode, in this case a concave disc of platinum or tungsten, is similar to that employed in the " pointolite " lamp in which a ball of tungsten is heated by cathode-ray bombardment (*see* Fig. 5(*a*)). As yet, however, no definite experimental data can be given.

(c) Electrostatic Deflecting Plates and Electromagnet.

In the oscillograph, shown in Fig. 2, the E.S. deflecting plates are mounted parallel to one another and to the axis of the tube, contact being made through side tubes of glass with spring contact fittings. The plates at present in use are 9 cm. long, 2 cm. wide and are situated a little over 1 cm. apart. For greater sensitivity, means are provided for bringing the plates closer together. Great care is taken to ensure that the electrostatic and magnetic fields are parallel, this being

indicated when the oscillograph records show that the independent deflections (E.S.) and magnetic are mutually at right angles.

In the present oscillograph the pole-pieces of the electromagnet are silver-soldered to the main tube of the oscillograph, whilst the external case of the magnet consists of laminated iron attached to the pole-pieces as shown in Fig. 2. In a more recent design (*see* Fig. 7) the whole magnet, including pole-pieces, is external to and free from the tube of the oscillograph, and is situated above the E.S. deflecting plates. The magnet is provided with a double-winding the purpose of which will be indicated later.

(*d*) *The Camera.*

The original design of this important part of the oscillograph provided for external electrical operation of the camera by means of suitable electromagnetic fittings on the camera itself. Experience showed, however, that these operations are more easily performed by means of conical plugs fitted through the wall of the camera vessel. The camera itself was at first designed to take either a single glass plate or a roll-film (V.P.K. size). Rotation of one of the plugs opens the door of the camera and exposes the film or plate. Rotation of the other plug in its socket winds the film from one spool to another as in the ordinary Kodak. The advantage of a roll-film is obvious, for 8 to 10 exposures can be made without " opening up " and re-evacuating the apparatus between successive exposures. The chief disadvantage of the film is, however, the moisture which it contains and which it is reluctant to release. If a film is used, it should be previously dried by storing in vacuo with P_2O_5 for a period of about 24 hours before it is required for actual oscillograph work. For another reason also, which will be evident when the question of photographic plates is discussed, the film may prove unsuitable. This is unfortunate, of course, for the great advantage of the film is its adaptability for a large number of exposures whilst occupying a very small space in the vacuum vessel. In the latest design of oscillograph (*see* Fig. 7), the camera accommodates six plates fitted on the faces of a hexagonal block of aluminium. A conical plug fitted on the outer casting of the camera chamber is employed to rotate the hexagon and thus expose any required plate to the action of the cathode rays. When an exposure is to be made, another plug removes the door of the camera, this door serving also as the phosphorescent screen, its upper surface being coated with a layer of phosphorescent material (willemite or calcium tungstate). The camera containing the six plates is in itself a light-tight unit which can be inserted in or removed from the oscillograph in daylight. By means of a hinged cover plate, light is prevented from entering through the viewing window when an exposure is being made. With such an arrangement six oscillograph records can be made before " opening up " the apparatus and admitting air. This is a great advance on the old design shown in Figs. 2 and 3, where it was necessary to re-admit air after each exposure when glass plates were being used. In the new design provision is also made for a second pair of electrostatic deflecting plates, arranged at right angles to the first pair, as an alternative to the use of the electromagnet in certain circumstances. All operations can be carried out in subdued daylight without the use of a dark room except for the purpose of developing exposed plates.*

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* Experiments have been made, with some degree of success, to render plates insensitive to light whilst retaining maximum sensitivity to cathode rays. With such plates, *all* operations, including development, can be carried out in subdued white light.

(e) Photographic Plates.   Absorption of Cathode Rays in Gelatine Films.

Reference has been made above to the possibility of increasing the photographic effect of the cathode rays by increasing the sensitiveness of the plates.   Now a plate which is most sensitive to ordinary light is not necessarily most sensitive to the action of cathode rays, for the factors involved in the two cases are of a distinctly different nature.   Early experiments indicated that the difference in photographic effect observed when plates varying in speed from 100 to 550 H. and D. when subjected to the same stream of cathode rays with exactly the same exposure, was too small to be of serious importance.   There appeared to be a slight advantage in favour of the faster plates, but this advantage was lost in the greater tendency of such plates to fog.   Consequently ordinary plates of speed about 150 H. and D. were usually employed.

The results were unsatisfactory, however, until a very powerful means of increasing the sensitivity to cathode rays was discovered, viz., the use of a " phosphorescent " photographic plate in December, 1920.*   An ordinary plate was dusted over with a thin layer of calcium tungstate and exposed in the usual manner to the cathode rays in the oscillograph.   The result obtained was much blacker than the usual faint record and indicated great possibilities in this method of increasing the photographic effect of the rays.   The photographic action of the rays is of course two-fold.   In the first place they produce their ordinary photographic action as they penetrate into the film, but more important still they produce a luminous effect in the phosphorescent material.   This luminous effect, particularly when calcium tungstate is used, has a powerful photographic action on the film, with which it is in close contact.

A simple calculation, easily confirmed by experiment, shows that slow cathode rays (p.d. 3,000 volts for example) such as those with which we are now dealing, will only penetrate extremely thin films of matter.   Thus it was found that the photographic effect of the rays after penetrating a layer of mica 0·001 in. thick was extremely small when compared with the direct effect on the bare photographic plate.   It thus seems probable that the rays are not sufficiently penetrating to get deep down into the gelatine of an ordinary plate—the energy being expended on the gelatine as well as on silver granules.   The phenomenon is exactly analogous to that of ultra-violet light incident on an ordinary gelatine plate.   It therefore seemed advisable to use plates containing a minimum quantity of gelatine,† so that the cathode rays might expend their energy almost entirely on the silver granules. With such a plate, made in addition phosphorescent by means of calcium tungstate (rich in ultra-violet phosphorescence), we should anticipate a greatly increased photographic effect from the cathode rays.   A quantity of calcium tungstate was therefore forwarded to Adam Hilger, Ltd., who very kindly supplied a number of ordinary Schumann plates and some specially prepared phosphorescent Schumann plates.

The results obtained with these plates have been very gratifying.   Both the ordinary Schumann plate and the phosphorescent variety are greatly superior to the

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* This method has also been employed by Levy, West & Baker (*see* Röntgen Soc. **Journ.**, 17, April, 1921), but the author was unaware of this work until a few months ago.

† Schumann plates for the ultra-violet possess this quality.

ordinary gelatine plate, yielding records of considerably greater intensity. The certainty of obtaining good records is now greatly increased, and there is less urgent need for obtaining increased intensity of the cathode ray pencil as indicated above. An examination of some of the records obtained with Schumann plates (untreated or phosphorescent) shows that a visible photographic impression is obtained when the **velocity** of the " spot " on the plate reaches 1,000 metres per second. There is no doubt that even this speed can be exceeded. With an ordinary gelatine plate, however, such a speed of recording would be quite impossible, for the records appear rather faint when the speed of the spot exceeds 50 metres per second. Examples
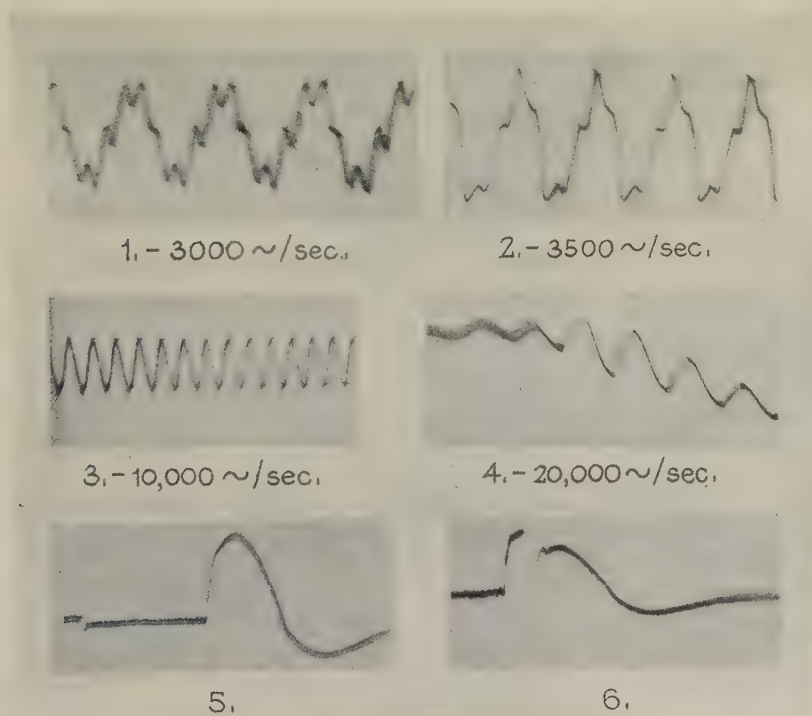
Fig. 6.

of records with Schumann plates—ordinary and phosphorescent—are reproduced in Fig. 6.

It appears from such observations that the most hopeful line of development of the cathode ray oscillograph for high-speed work lies in the direction of increasing the sensitiveness of the photographic plate, rather than in attempting to obtain further increases in intensity of the cathode ray beam. Investigations are in progress to improve the oscillograph in both these directions.

The importance of the use of Schumann plates in recording by means of low-gress to voltage cathode rays cannot be overestimated.

(f) Generation of Time Axis of Oscillograph Record.

In the majority of high-frequency electrical problems we are concerned mainly with the variation of current or voltage with time. The generation of a time axis in the cathode ray record is therefore of considerable importance. This can be accomplished by mechanical, electrostatic or electromagnetic means.

(i) *Mechanical Methods.*—This is the most obvious method. Here it is required that the plate or film is moved at a definite rate in a line at right angles to the direction of deflection of the cathode rays by the A.C. supply. For example, if the film were attached to the surface of a drum inside the vacuum, the drum could be driven either directly through a gland, magnetically or as an induction motor. If magnetic or electrical methods of driving are used, however, it is necessary that stray fields should be very weak (insufficient to affect the cathode ray beam) or that they should be cut off at the time of making a record, the drum in this case continuing to rotate by virtue of its own inertia. This latter method is the most promising but is not in actual use since other and simpler methods serve the same purpose equally well—especially when glass plates are used.

There is a possibility also of removing the film *outside* the vacuum and displacing it by the usual mechanical methods, but this method presents serious difficulties owing to the absorption and scattering of the cathode beam in passing from the vacuum through a thin window (mica or similar material) to the photographic film.

(ii) *Electrostatic and Electromagnetic Methods.*—(a) A simple method of generating a time-axis particularly applicable in recording impulsive or non-periodic phenomena is that used by D. A. Keys. The method employs sinusoidal alternating current or voltage of known frequency to excite the electro-magnet or the electrostatic deflecting plates—the cathode rays being thereby deflected sinusoidally. The impulsive effect to be recorded deflects the rays at right angles to the track traced in the above manner.

(b) The disadvantages of the A.C. method just outlined are obvious and the method is quite useless in recording A.C. wave-forms or other types of periodic oscillations. The following method, suggested by Dr. C. V. Drysdale and used also by M. Dufour, has proved much more satisfactory for this purpose. The method depends for its success on the finite rate of rise of current in an inductive circuit* when a p.d. is applied. In the early stages of rise of current the relation between current and time is practically linear. Hence if such a rising current is passed through the electro-magnet of the oscillograph, the deflection of the spot at any instant will be proportional to current and consequently to time.

In the practical application of this method it is found advantageous to employ an auxiliary winding to the deflecting electro-magnet, this winding carrying a steady current which deflects the spot off the plate in a direction opposite to that produced by the primary winding. The current through the latter is, however, sufficiently powerful to overcome the effects of that in the auxiliary winding and thus produce a much greater deflection of the spot in the opposite direction. Hence, on closing the main magnet circuit the spot is drawn from a position just off one edge of the plate to a position well off the other edge, *i.e.*, the spot trails across the plate from one edge to the other at a definite rate depending on L and R, and leaving a record

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* Analogous considerations apply also in the case of the charging or discharging of a condenser.

of the A.C. wave-form which is being studied. Records obtained in this manner are quite satisfactory, as will be seen from the specimens reproduced in Fig. 6.

The same result as that obtained in the "inductance" method can, of course, be obtained more directly by varying the current through the electro-magnet or the p.d. across the deflecting plates in a manner proportionate to time. This may be done by means of a simple mechanical method of continuously varying the resistance of a rheostat—it being essential, of course, that no intermittent variation producing sudden "jumps" in current are permitted. A rotating potentiometer has served the purpose extremely well, making it possible to vary the rate of traverse of the spot across the plate over a very wide range of speeds. Using such a device records of impulsive and periodic phenomena have been obtained with a linear time scale.

IV. EXPERIMENTAL RESULTS.

A number of typical records are reproduced in Fig. 6.

Nos. 1 *to* 4 are records of A.C. wave-forms, the source of A.C. being a valve-oscillator. Records of this character have been obtained up to a fundamental frequency of 50,000 ∿/per sec., this upper limit being fixed by the presence of the iron-core in the deflecting electro-magnet. In the new design of oscillograph at present in hand, this defect will be eliminated and it is anticipated that records at much higher frequencies will then be possible. It will be observed that some of the A.C. records are far from sinusoidal. Such records suggest interesting possibilities in the study of oscillations in valve circuits.

Nos. 5 *and* 6 are records of impulsive phenomena, representing the back e.m.f. at the break of current flowing in a coil of large inductance. The time-scale in this case was generated by the "inductance" method, and an automatic timing device, which it is unnecessary to describe here, was employed to ensure that the "break" occurred at the correct instant when the cathode rays were crossing the photographic plate. The time of the first semi-oscillation in this record is about 0·001 second. A potentiometer method of voltage-reduction was employed, the maximum deflection on the record corresponding to a back e.m.f. of about 6,000 volts on the coil. Record No. 6 is of a similar type, showing the result of a defect in the commutator which breaks the current in the main circuit.

V. CONCLUSION.

New Design of Oscillograph.

From what has been said above it will be realised that a commercial form of low-voltage oscillograph has tremendous possibilities in its applications to research and to electrical engineering problems. The satisfactory development of a sensitive high-frequency oscillograph opens up a wide field of investigation in wireless telegraphy. An attempt is also being made to develop a sealed form of cathode ray oscillograph with external photographic arrangements. For this purpose experiments are being made with various focusing devices, so as to obtain an intense beam of cathode rays which will give an appreciable photographic effect after passage through a "window."

With the knowledge gained in the use of the oscillograph described above a new oscillograph has been designed which it is hoped will remedy the defects of the earlier patterns.

Ebonite Cap.

Spiral Filament.
Tungsten or Platinum.
(Lime Coated.)

Glass Bulb.

Pinhole Tube.

Electro-magnet.
(Two Windings.)

E.S. Deflecting Plates.

Ebonite

Glass

Electrodes of
Deflecting Plates.

Brass Tube.

Second Pair of
Deflecting Plates.

Drying Bulb.

Pumping Tube.

Air-tight
Viewing Hole
on Body Front
not shown.

Shutter Open
Shown Dotted
in Closed Pos-
ition.

Conical Plug to
Open Door of
Camera.

Sensitive
Photo-Plate.

Plug to Rotate
Hexagonal Plate-Holder

Removable
Inner Case.

Brass Body

Drum
carrying
6 Plates.

FIG. 7.

In conclusion, I should like to take this opportunity of thanking the Admiralty for permission to publish this Paper, Dr. C. V. Drysdale for his constant interest and for many useful suggestions in the design of the oscillograph, and Mr. E. H. Lakey, B.Sc., for his invaluable assistance in obtaining the oscillograph records.

SUMMARY.

A description is given of a new form of Cathode-ray oscillograph of a pattern adapted for commercial production and laboratory use.

Details are given of the principal features of the instrument which is considerably more sensitive than the Dufour oscillograph. The increased sensitiveness is due to the production of the cathode rays at a comparatively low voltage, viz., 3,000 volts, whereas the Dufour instrument employs 60,000 volts.

Various methods are described for focusing the cathode ray stream, and for increasing the sensitiveness of the photographic plates. The best results have been obtained with Schumann plates containing a proportion of calcium tungstate. This material phosphoresces with a light rich in ultra-violet, and consequently the secondary luminous effect on the Schumann plate is very great.

Mechanical, electrostatic and electromagnetic methods are described for generating a time axis on the records. For certain purposes this time-axis is sinusoidal, whilst for others it is linear.

Typical records of high-frequency A.C. wave-forms and of impulsive electrical phenomena are reproduced in illustration.

In conclusion, brief reference is made to the wide applications of the low-voltage type of oscillograph and to a new design of oscillograph.

DISCUSSION.

Mr. MINES inquired whether any measurement had been made of the potential gradient in the tube with a view to determining the distribution of potential along the path of the rays. Was anything known of the current density of the rays, and was there (as in the arc lamp) a limiting value to such current density, involving a maximum intensity for the spot of light beyond which it would be impossible to go? And had the Author tried the focussing effect of charging the cylindrical shield, as at (d), Fig. 5, to a positive potential of, say, 100 volts?

Mr. R. A. WATSON WATT (partly communicated): I would join in congratulating Dr. Wood on his very lucid account of a most important and valuable instrument of research. Perhaps the most outstanding feature of his work is the way in which the oscillograph has been converted into a robust engineering structure, which the " glass shy " engineer can approach with more confidence than he feels towards the older forms of blown glass tube, and into an instrument with a much higher expectation of life than these older forms possessed.

In considering cathode ray oscillographs in general, one should not lose sight of the fact that there is a wide field of application for sensitive tubes in which it is not necessary to resort to photography with its attendant difficulties and complications. Single axis working, in which the tube is valuable as a direct reading high-frequency voltmeter, is in itself sufficient justification for such tubes. But Dufour has shown that in wave form work one can obtain visual images of radio frequency wave forms from his high-voltage tubes, while recently, by the courtesy of the International Western Electric Co., I have been able to satisfy myself that a reliable visual image of a transient deflection of 10^{-4} sec. duration can be obtained with the 300-volt beam of their tube. The fluorescent spot had a translational speed of about 5×10^{4} cm. per sec. in this experiment. This result, whether due to persistence of vision or to persistence of fluorescence, is much more satisfactory than one had the right, a priori, to expect.

On minor matters of detail it might be desirable in a Paper which will take an important place in the chronology of cathode ray oscillographs that the Author should make it clear that he does not assign priority for the introduction of the hot cathode, of the A.C. time base, and of the D.C. time base, to the workers referred to in this Paper. The Author's own familiarity with the history of the subject might lead to the appearance of neglecting the priority of Wehnelt (1906)

as to the hot cathode, of Fleming (1913) as to the A.C. time base, and of Dufour (1914) as to the D.C. time base, which he superposed on the A.C. base, so that the final image is based on the plane projection of a cylindrical helix.

I should be grateful if Dr. Wood could say whether any data are available as to a tube working on the 100 volts, which he mentions as a lower limit. A tube of this extreme sensitivity would be most valuable, but I know of no references to a stably operating tube with an accelerating voltage below 300 volts.

Finally, I should like to ask the Author whether he has any experience of Van der Bijl's " positive ion " method of focussing, as used in the Western Electric tube. It appears to be a method of great value, particularly in view of the fact that the commonly used electro-magnetic " paralleling " method reduces the sensitivity of the tube. My experience of low-voltage tubes leads me to conclude that at the present moment the limitation of sensitivity for visual observation lies, not in the mechanism of beam production or concentration, but in the fluorescent materials used for the screen, and that progress in that direction can still be made.

Mr. R. S. WHIPPLE said that the beautiful instrument described would meet a real demand, as oscillographs using a cold cathode are cumbersome and costly, while the one under discussion should be within the means of most laboratories. The method of phosphorescent sensitisation opens up great possibilities. The Paper is written in such a way as to be unusually readable.

AUTHOR'S reply (communicated) : In reply to Mr. MINES : No measurements have been made to determine the potential gradient along the path of the rays in the bulb. There is, of course, no further change of potential after the rays pass through the pinhole tube, and enter the deflecting chamber. The *total* current between the hot cathode and the anode is usually of the order of 1 milliampere. No direct measurements have been made of the current density of the cathode stream after passing through the pinhole tube. This could easily be done of course by a capacity-electrometer method. There is no evidence to indicate that a maximum intensity of the spot has yet been reached—the brilliancy apparently increases up to the point of breakdown of the bulb or of fusion of the filament.

With regard to focusing by means of a cylindrical shield and central wire as shown in Fig. 5d, it is proposed to use a potentiometer device which will give a continuous voltage variation over any required range, thus providing a simple means of adjusting the position of focus of the rays.

In reply to Mr. WATSON WATT : The Wehnelt hot lime cathode is now so well known that I consider it unnecessary to assign priority to the inventor. Lime and thoria-coated filaments of platinum and tungsten are now in everyday use in thermionic valve manufacture.

I was not previously aware that the A.C. time base was first proposed by Fleming. In any case the D.C. time base is preferable whether it is generated by the inductance method or by the rotating potentiometer.

With suitable vacuum conditions (rather a soft bulb), a beam, sufficiently powerful for visual observations, can be obtained with a p.d. as low as 100 volts ; but, as I stated in the Paper, considerable difficulty is experienced in obtaining a sufficiently intense beam for photographic purposes if the generating voltage falls below 1,000. If it is essential for certain purposes to use low voltage rays then the photographic action might be increased by further acceleration of the rays subsequent to deflection. For ordinary purposes, however, where a good photographic record is to be obtained, 3,000 volts generating potential has been found quite satisfactory. Other methods of increasing sensitivity are at present under consideration ; these, however, will form the subject of a future Paper.

It should be remembered that cathode rays generated at low voltages, *e.g.*, 300 volts, are so much reduced in energy that they cannot penetrate sufficiently deep into the phosphorescent screen or the photographic film, hence the observed effects with such rays are always very much less than with rays at higher voltages. Since the square of the velocity of the rays is proportional to the applied voltage it might be assumed that the energy available to produce photographic or phosphorescent action is proportional to the latter. That is, the photographic effects of rays generated at 30,000, 3,000 and 300 volts would be in the ratio 100, 10 and 1 respectively.

It would seem on these grounds that improved visual or photographic effects can best be obtained by using screens or films of less dense material than such as is at present employed.

I am not familiar with Van der Bijl's " positive ion " method, and should be grateful to Mr. Watson-Watt for information.*

* *See* p. 127.

DEMONSTRATION *of Some Applications of the Gyroscope.* By M. PAUL
SCHILOWSKY, *Chairman of the Gyroscopic Society, Petrograd.*

M. SCHILOWSKY said that the principles of the gyroscope are insufficiently
taught in technical colleges, with the result that the subject has received very little
attention from the engineering profession. Engineers are in consequence unduly
sceptical as to the applicability of such principles to practical problems, although
a staff of some 50 Russian experts under the speaker's chairmanship had evolved
designs for the construction of a perfectly safe and practicable monorail system
for a suburb of Petrograd. The models he was exhibiting were not to scale. In
practice the gyrostatic apparatus would form from 3 per cent. to 5 per cent. of the
load of a ship, and from 5 per cent. to 10 per cent. of the load of a monorail carriage.
The underlying principle of his inventions is as follows : To stabilise a system in
unstable equilibrium, a reaction must be set up between the system and the gyrostat
of such a character as to *help* the precession of the gyrostat during the return of the
system to normal. To check the oscillations of a stable system, the reaction must
be such as to *oppose* such precession. The gyrostat must, of course, be power-driven
to neutralise friction. The exhibits included the following :—

(*a*) A collection of apparatus for teaching purposes, comprising *inter alia* models
illustrating the precession of the earth, a method of optically projecting an image
of a spinning top, and small monorail models. The above-mentioned principle was
illustrated by means of a top which, when spinning on a sharp point, exhibited
precession. When, however, the point was furnished with a spherical knob, the top
" slept " until its angular momentum was exhausted by friction, when it abruptly
collapsed.

(*b*) Model of a rocking ship. The gyrostatic flywheel is mounted with its axle
vertical in a frame, which can both rock about and slide along an axis transverse
to the ship. When the ship rolls to starboard the axle of the wheel tilts in precession
in a fore-and-aft plane, and at the same time the gyrostat slides bodily sideways under
gravity, its frame engaging a rack which holds it with the axle in the tilted position.
When the ship begins to roll back, the axle tries to reverse its fore-and-aft tilt, but
is prevented by the engagement of the rack, which is spring-controlled. Thus
precession is opposed during the return of the ship to normal and (the equilibrium
being stable) rolling is checked by the resulting reaction.

(*c*) Model of an aeroplane carrying a gyrostat. The problem of combining
automatic stability with mobility while avoiding dangerous stresses was discussed.

(*d*) Models of monorail gyrostatic apparatus. In the most recent design the
flywheel is mounted with its axle vertical in a frame which can tilt in a fore-and-aft
plane, and also slide sideways under gravity. The frame is surmounted by a pinion
coaxial with and geared down from the flywheel ; and the pinion lies between, but
normally clear of, two parallel fixed racks mounted on the carriage, and having their
lengths in a fore-and-aft direction. When the carriage tilts to one side, the axle
of the flywheel tilts forward or backward in precession, and at the same time the
gyrostat slides bodily sideways, so that the pinion engages one or other of the two
fixed racks. The pinion then climbs back along the rack to its normal position—
i.e., in the same direction as the precession which it would exhibit during the righting
of the carriage. Thus such precession is aided and (the equilibrium being unstable)
the resulting reaction restores the carriage to the upright position.

(*e*) Angular-velocity indicator for aeroplanes.   Angular velocity of the aeroplane about a vertical axis causes a tendency to precess in a gyroscope rotating about an horizontal axis.   This tendency is balanced by a gravity control, and the angle moved through in attaining a balance is indicated by a pointer, affording a measure of the required angular velocity.

DISCUSSION.

Sir GEORGE GREENHILL expressed the indebtedness of the meeting to M. Schilowsky for his brilliant experiments and address.

Prof. C. VERNON BOYS expressed his pleasure at seeing a Russian scientist alive and well. He hoped M. Schilowsky would take back to his confrères in Petrograd a message of goodwill and sympathy. He admired the lecture-models, especially those illustrating the earth's precession. A monorail carriage for passengers seemed at first sight impracticable, but the designs which had been actually worked out must be accepted as evidence to the contrary. Such a system would effect great economies, and it would much improve the comfort of travelling by removing the constraint of a double rail, which is responsible for the shocks experienced in an ordinary railway carriage. In the long coach with a four-wheel bogey at each end, of which a model was shown, pitching as well as lateral movements should be eliminated. An example of the opposite extreme is afforded by the so-called mono-railway between Listowel and Ballybunion, which is monorail only in the Irish sense since it has five rails designed to give a maximum of constraint. The result is that the noise and shaking are excessive, a speed exceeding 15 miles an hour would probably destroy the track, and the design is such that the driver cannot cross from one side of the engine to the other without stopping the train. The designs of M. Schilowsky, on the other hand, aim at a minimum of constraint, with correspondingly superior results. The problem of replacing existing systems by the monorail, however, would involve questions of finance, and of vested interests which would defy any calculus known to science. One difficulty occurred to him as serious. It was clear that the carriage would run on a straight or uniformly curved track ; but what would happen in the case of a track whose curvature was changing ?

Sir JAMES HENDERSON expressed admiration for the experiments, and the skill and ingenuity of the practical designs. He fully agreed as to the necessity for educational propaganda, and had, in fact, introduced a course on the subject into the Royal Naval College some 17 years ago. The apparent fact that a mono-railway had been constructed in Petrograd was one of much interest.

M. SCHILOWSKY, in reply, said that no difficulty was experienced in negotiating short curves. (This fact was illustrated by means of a small model running on a curved track, and also on a turntable.) Long curves presented certain difficulties, which could not be discussed fully in the time available. In principle such difficulties are met by loading the instability of the system. Unfortunately the impression that the Russian Government had built a railway was imperfectly true. Some 10 kilometres of track had been constructed, but the scheme had got no further.

DEMONSTRATION *of a Low Voltage Cathode Ray Oscillograph.* By Mr. R.
WEBB, *International Western Electric Company.*

THIS instrument, which is designed to work at 300 volts, is intended for manufacture
on a commercial basis. The cathode consists of a hot platinum filament coated with
certain oxides, and formed into a circle coaxial with the path of the rays. It is
protected from bombardment by positive rays, which would disintegrate it, by a
screen in which is cut a circular hole slightly less in diameter than the filament,
and has a life of about 200 hours. The anode is a platinum tube through which the
rays pass. The deflecting fields are electrostatic and are provided by two pairs
of plates at right angles, but electro-magnetic deflections may be produced with the
help of a pair of small coils fitted outside the tube. The bulb is in the form of a
conical flask, the cathode being at the narrow end, so that the rays impinge on the
flat bottom, which is coated inside with fluorescent matter, and the luminous trace
of the rays can be seen from outside through the bottom of the flask. The spot
can be focused by adjusting the filament current. This is brought about with the
help of the gas in the tube which is ionised by the rays.* The apparatus was con-
nected up† to show the characteristic of an oscillation-generating valve, so that the
potential difference between one pair of plates was proportional to the grid potential,
and that between the other pair to the plate current. The resulting trace consisted
of a single curved line terminating in a loop of peculiar form at one end.

DISCUSSION.

From Mr. R. A. WATSON WATT (communicated) : The method of focusing suggested by
Van der Bijl, and applied by J. B. Johnson, in the very beautiful tube demonstrated at this
meeting, employs positive ions derived from the residual gas, actually argon at about 0·005 mm.
mercury pressure, to neutralise the spreading of the electron beam due to mutual repulsion and
to initial divergence. The impact of the electrons in the beam on molecules in its path produces
ionisation, and the relatively low mobility of the heavy positive ions results in a concentration
of positive electricity along the path of the beam. The space charge of random electrons outside
the beam aids the production of a radial field opposing divergence. The number of positive ions
along the path of the beam depends on the electron current, which, under working conditions,
is limited by the cathode temperature. The strength of the radial field, and consequently the focal
distance of the beam, can thus be controlled by the rheostat governing the cathode heating current.
Experimentally Johnson finds that a radial field of the order of 3 volts per centimetre is required
in the type of tube which has been standardised for production. In practice one finds that
a change of 5 per cent. in filament current reduces the diameter of the fluorescent spot (at a
screen distance of 20 cm.) from about 5 mm. to less than 1 mm., so that the focal control is very
sharp and sensitive. The presence of so many positive ions would, in the case of the older forms
of tube, have led to very short tube life by destructive bombardment of the filament. Johnson
minimises this trouble by making the free volume of the inter-electrode space less than 1 cubic cm.,
and by shaping the filament so that it is out of the " line of fire " of ions shooting through the
canalising anode. A filament life of 200 hours is thus obtained. Reference may be made to
J. B. Johnson's Paper on " A Low Voltage Cathode Ray Oscillograph " in the Journal of the
Optical Society of America and Review of Scientific Instruments, Vol. 6, pp. 701-12, Sept.,
1922.

* J. B. Johnson, Journal of the Optical Society of America and Review of Sci. Instruments.
September (1922).
† E. V. Appleton and B. van der Pol, Phil. Mag., August (1921).

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